PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

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THIRTY-FIFTH SESSION, 1898–99
(since the Formation of the Society, January 16th, 1865).

November 10th, 1898.

THE FIFTH ANNUAL GENERAL MEETING OF THE LONDON MATHEMATICAL SOCIETY, as incorporated under the Companies Act, 1867, on October 23rd, 1894, held at 22 Albemarle Street, W.

Prof. E. B. ELLIOTT, M.A., F.R.S., President, in the Chair.

Present, twenty members.

The President briefly referred to the recent deaths of Mr. R. Holmes, formerly a member of the Society, and for a short time its Honorary Librarian; Mr. Walter Wren, elected October 16th, 1865; and Dr. J. Hopkinson, F.R.S., elected February 13th, 1873.

The Treasurer, Dr. J. Larmor, gave a short abstract of his report. Its reception was moved by Mr. A. B. Kempe, and seconded by Mr. S. Roberts, and carried.

The Chairman announced that Mr. Gallop would be asked again to act as Auditor.

Mr. Love made the following statement:—Number of foreign members, 13+2 new = 15; number of ordinary members, 228+10 new = 238; one foreign member and four ordinary members had died, and one member had resigned; so that the number of ordinary members was 233.

The Society's losses by death, had been, in addition to those stated above, Signor Brioschi, elected honorary member May 9th, 1878; Dr. Percival Frost, elected December 9th, 1869; and Mr. Henry Perigal, elected January 23rd, 1868.

The exchanges of *Proceedings*, and journals purchased, remain unaltered.

The following communications had been made or received:—

Note on a Property of Pfaffians: Mr. H. F. Baker.

The Conformal Representation of a Pentagon on a Half Plane: Miss M. E. Barwell.

On the General Theory of Stationary Motion in an Infinite System of Molecules: Mr. S. H. Burbury.

The Construction of the Straight Line joining Two given Points; On the Continuous Group that is defined by any given Group of Finite Order (two papers); and on Linear Homogeneous Continuous Groups whose Operations are Permutable: Prof. W. Burnside.

On a Law of Combination of Operators (Second Paper); and Transformations which leave the Lengths of Arcs on Surfaces unaltered: Mr. J. E. Campbell.

On Aurifeuillians (with a supplement): Lt.-Col. A. J. C. Cunningham.

On the Numerical Value of $\int_0^h e^{x^2}dx$: Mr H. G. Dawson.

The Transformation of Linear Partial Di erential Operators by Extended Linear Continuous Groups: Prof. Elliott.

The Character of the General Integral of Partial Differential Equations; On those Transformations of the Coordinates which lead to New Solutions of Laplace's Equation; and An Essay towards the Generating Functions of Ternariants: Prof. Forsyth.

A Theorem concerning the Special Systems of Point-Groups on a Particular Type of Base Curve: Miss F. Hardcastle.

The Integral $\int P_n^2 dx$, and Allied Forms in Legendre's Functions, between Arbitrary Limits: Mr. R. Hargreaves.

On Discontinuous Fluid Motions involving Sources and Vortices: Mr. B. Hopkinson.

On the Reflection and Transmission of Electric Waves by a Metallic Grating: Prof. H. Lamb.

On the General Theory of Anharmonics: Prof. E. O. Lovett.

Point-Groups in a Plane, and their effect in determining Algebraic Curves: Mr. F. S. Macaulay.

On the Calculus of Equivalent Statements (Seventh and Eighth Papers): Mr. H. MacColl.

Note on Bessel Functions; and Zeroes of the Bessel Functions: Mr. H. M. Macdonald.

On the Poncelet Polygons of a Limaçon; and On a Regular Rectangular Configuration of Ten Lines: Prof. F. Morley.

On certain Regular Polygons in Modular Network: Prof. L. J. Rogers.

On the Calculation of the most Probable Values of Frequency-Constants, for Data arranged according to Equidistant Divisions of a Scale: Mr. W. F. Sheppard.

On the Intersections of Two Conics of a given Type; and On the Intersections of Two Cubics: Mr. H. M. Taylor.

The Geodesic Geometry of Surfaces in non-Euclidean Space: Mr. A. N. Whitehead.

On Systems of One-Vectors in Space of n Dimensions; Mr. W. H. Young.

Mr. G. Heppel and Mr. J. B. Dale having been appointed Scrutators, the ballot was taken, with the result that the following gentlemen, nominated by the Council, were elected to serve on the Council for the ensuing Session:—The Rt. Hon. Lord Kelvin, G.C.V.O., F.R.S., President; Prof. Elliott, F.R.S., Prof. H. Lamb, F.R.S., and Lt.-Col. A. J. C. Cunningham, R.E., Vice-Presidents; Dr. J. Larmor, F.R.S., Treasurer; Mr. R. Tucker and Mr. A. E. H. Love, F.R.S., Hon. Secretaries. Other Members of the Council:—Dr. Glaisher, F.R.S., Prof. M. J. M. Hill, F.R.S., Dr. Hobson, F.R.S., Prof. W. H. H. Hudson, Mr. A. B. Kempe, F.R.S., Dr. F. S. Macaulay, Major MacMahon, R.A., F.R.S., Mr. D. B. Mair, and Mr. W. D. Niven, C.B., F.R.S.

Lt.-Col. Cunningham was then moved to the Chair, and, after moving a vote of thanks to Prof. Elliott for his conduct during his two years' tenure of office, which was carried by acclamation, called upon him to read his Presidential Address. After the reading of the Address, Prof Elliott consented, in reply to the wish of the members present, to its being printed in the *Proceedings.*

The following papers were formally communicated:—

The Structure of certain Linear Groups with Quadratic Invariants: Dr. L. E. Dickson.

Multiform Solutions of certain Differential Equations of Physical Mathematics and their Applications: Mr. H. S. Carslaw.

On the Null-Spaces of a One-System and its Associated Complexes: Mr. W. H. Young.

On the Functions Y and Z which satisfy the Identity $4(x^p-1)/(x-1) = Y^2 \pm pZ^2$: Prof. L. J. Rogers.

The following presents were received for the Library:—

"Journal of the Institute of Actuaries," Vol. xxxiv., Pt. 3 (two copies); October, 1898.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xxii., St. 9, 10; Leipzig, 1898.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zurich," 1898, 2[es], 3[es] Heft.

"Proceedings of the Cambridge Philosophical Society," Vol. IX., Pt. 9; Easter, 1898.

"Jornal de Sciencias Mathematicas e Astronomicas," pelo Dr. F. G. Teixeira, Vol. XIII., No. 4; Coimbra, 1898.

"Monatshefte für Mathematik und Physik," Jahrgang IX., 1898, Pt. 4; Wien.

"Bulletin of the American Mathematical Society," Vol. V., No. 1; New York, October, 1898.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," 1898, Heft 3.

"Mémoires de la Société des Sciences de Bordeaux," Tome III., Cahier 1; Paris, 1898.

Roe, E. Drake.—Die Entwickelung der Sylvester'scheu Determinante nach Normal-Formen," 8vo; Leipzig, 1898.

"Bulletin de la Société Mathématique de France," Tome XXVI., Nos. 7, 9; Paris, 1898.

"Mathematische Annalen—General Register zu den Bänden I.–L.," von A. Sommerfeld; Leipzig, 1898.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. VII., Fasc. 7, 8, Sem. 2; Roma, 1898.

"Annales de la Faculté des Sciences de Toulouse," Tome XII., Fasc. 4; Paris, 1898.

"Transactions of the Cambridge Philosophical Society," Vol. XVII., Pt. 1.

"Annali di Matematica," Serie 3, Tomo I., Fasc. 4; Milano, 1898.

"Educational Times," November, 1898.

"Journal de l'Ecole Polytechnique," Série II., Cahier 3; Paris, 1897.

"Annals of Mathematics," Vol. XII., Nos. 4, 5; Virginia, 1898.

"Indian Engineering," Vol. XXIV., Nos. 13–16, Sept. 24–Oct. 15, 1898.

"Atti del R. Istituto Veneto," Tomo LV., Dispense 3–10; Tomo LVI., Dispense 1–7.

"Philosophical Transactions of the Royal Society of London," Vols. CLXXXVII.–CXC.

*Some Secondary Needs and Opportunities of English Mathematicians.** *By* E. B. ELLIOTT. Read November 10th, 1898.

MR. VICE-PRESIDENT,—In surrendering the duties which I have had the honour of being entrusted with to one who confers the greatest possible honour on the Society by consenting to undertake them, I reflect with satisfaction that the two years of my occupancy of the chair have been years of fruitful energy on the part of our members, and of real achievement in many a field of mathematical effort. The two volumes of our *Proceedings* which cover the period, of which the second is not yet entirely in print, though unusually far advanced, will, it is probable, prove to be together greater in bulk than those produced during any previous Presidency, and, in wideness of subject-matter and genuine scientific interest, are not out of keeping with their extent, but are a just cause of pride to the Society. I will refrain from dwelling upon the importance of, and the promise afforded by, particular contributions to these volumes by present members of the Society; but feel bound at once to congratulate mathematicians on the fact that the Society saw its way, after listening two years ago to the masterly discourse of my immediate predecessor in the Presidential chair upon the "Combinatory Analysis," to publish, as the late Professor Sylvester's life was drawing to its close, that profound thinker's "Outlines of Seven Lectures on the Partitions of Numbers." This remarkable syllabus of proved results and teeming ideas had lain almost inaccessible, though privately printed, for thirty-seven years. Mathematicians now have it in full view, and are in position to profit from the beauty of the truth it exhibits, to strive to develop what it sketches, to grapple with its unsolved puzzles, and get nearer to deeply hidden secrets round which the mighty brain of the author was working. Had our Volume XXVIII. contained nothing but these outlines, and the Presidential Address to which their publication was a sequel, instead of being otherwise most full and worthy, our Society would not that year have lived in vain.

Losses by death in the two years have been heavy. The veterans

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* Address delivered on retiring from the office of President.

of mathematics have continued to pass away; and there are gaps in the ranks of the stalwart. On February 19th, 1897, the life of Karl Weierstrass ended—a long life, a full, and a magnificently fruitful one. He stood supreme among modern analysts. Late in life he was happily induced to begin the collected publication of the works of his genius. The completion of this publication is now left to others; but he lived long enough to see rich harvests gathered where he had wrought, long enough for it to be a fact that he saw his own writings classical and his name historic. Another eminent Continental mathematician, whose loss early this year we have especial reason to mourn, was Francesco Brioschi, President of the Accademia dei Lincei, an honorary member of our Society. He was a man whose contributions to mathematical knowledge appeal to us all the more as there was a certain kinship between him and the English school of researchers in higher algebra. This kinship was only one among other marks of a breadth of scientific interest, which in his young days was novel in Italy, of enthusiastic participation in investigations whose first home lay beyond the limits of his own country, and its most natural schoolmistress France. Thus his first labours bore fruit of lasting national importance in removing the defect of narrowness from the school of mathematics in which he had been brought up. His countrymen revere him as one who roused mathematical energy and reformed mathematical education among them. His work for education in his own land was not, however, in the domain of mathematics only; and his work for mathematics was very far indeed from being only of national importance.

The death of Brioschi diminished a second time the number of our foreign honorary members. At our last meeting we strengthened our roll by adding to it the distinguished names of Dr. H. A. Lorentz and M. Emile Picard.

Our home list of ordinary members has suffered the loss of a number of distinguished, and some venerable, names. The supplements to our *Proceedings* contain, or will contain, brief tributes to the friends who have passed from us; but I must here just allude to some of them: to Mr. Perigal, whose years exceeded those of any other member, whose interest in science was still young almost till his death in his ninety-eighth year, who used at our meetings in the young days of the Society to help it in one of its aims—not so much, I regretfully realize, to the fore now as was the case then—of making our gatherings together occasions for informal interchange of ideas, and consequent stimulation of the comparatively unlearned among those who

had the cause of mathematics at heart; to Dr. Frost, who so long and so successfully was to the front in didactic mathematics, which it is also, I trust, one of the aims of our Society to foster and improve, whose admirable text-books, occupying, as they do, a middle place between those which restate elementary matters for junior teaching and those whose ambitious aim it is to lay before the higher student guides to all that has been done in particular departments of investigation, present themselves to me as models to be imitated with high advantage in various regions of modern mathematics; to Lt.-Col. Campbell, one of those who most realized that our Society was doing a good work well, who served it not with his presence on the Council only, but with his means, who figures as our second great pecuniary benefactor, and whose generous gift of £500, untrammelled by any stipulation as to application—such was the trust he had in the management of the Society and the fixedness of its scientific purpose —has been helpful in keeping us so far from having to consider whether a limit must not be placed to the amount of worthy matter which our funds will allow us to publish. I have kept till last the revered name which is uppermost in my own mind when I think of the past heroes of the Society. On the 15th of March, 1897—two years after Cayley and less than a month after Weierstrass—died James Joseph Sylvester. Which of us has not sat fascinated by his eye, as, full of fire, it fell upon us, from this spot it may have been; which of us has not listened, almost entranced, to his living words? Who of us all has not seen, for the time at least, almost majestic beauty in a minor mathematical truth as his enthusiasm has displayed it before us; who has not felt the profound almost within his grasp as Sylvester has pointed the way? The very blemishes which a cold critic might say mar the permanent usefulness of his brilliant writings, which taxed editors and printers to near the limits of their endurance, the impatient fitfulness, the restless haste of production, the supplements and restatements, the digressions from which there was no return, were but signs of a vital force to which it is hard to find a parallel, a vital force which vivified the energies of all who came under its direct influence. I rejoice that a visible memorial is to be raised to commemorate Sylvester's genius. I could wish that within our own Society some lasting tribute to his memory, coupled with Cayley's, had been organized, that the names of our second and third great Presidents might always be with us, even as that of our first President, De Morgan. I could wish even more that the worthiest of memorials could be raised to him by the collected

publication of all his contributions to the existing sum of mathematical thought, with their idiosyncrasies, their marks of his unique personality, still upon them as they came from his pen, or, at any rate, as they left the perplexed compositor. I regret exceedingly that he could not in his last years, like Cayley, like Weierstrass, avail himself of an opportunity, which, I believe, occurred to himself, collect the matter and begin the issue. It may be only a dream that funds for the publication may yet be found. But, even if his writings remain scattered, his memory will long be green among us. His achievements were in men wherever he went, even more than in memoirs wherever he sent them. In not a few companies of English-speaking mathematicians it may be said of Sylvester—" Si monumentum requiris, circumspice."

A mathematical event of the last two years has been the arranging for and holding in the summer of 1897 of an International Congress of Mathematicians at Zürich. On this subject, so dear to his heart, I should like to have been this evening listening to the one of our former Presidents, who, in 1892, addressed us on " Collaboration in Mathematics." He could tell us of the steps taken by himself, and others, to bring about the desired reunion; he could tell us how far his hopes were realized, and of the lessons learned and profit gained by interchange of ideas and social intercourse between mathematicians of one nation and another; also of the prospects of frequent recurrence in the future of such opportunities for conference, to the mutual benefit of all concerned. On this first occasion a few members of our Society participated, though not so many as would have been the case but for the simultaneous meeting of the British Association in Canada. On another occasion, we trust that English mathematicians will be numerously, as well as strongly, represented.

In connexion with this subject of international scientific collaboration, two other matters occur to me for mention. One, in which mathematicians as well as other scientific workers are interested, is the present active devotion on the part of a Committee working under the auspices of the Royal Society of much care and thought to the preparation of a scheme for an International Catalogue of Scientific Literature, published throughout the world, that workers in any line of research may have means of readily knowing where to go to see what is being done by others in the same field. Our Council has been taken into consultation as to the interests of mathematics, and —well off as we mathematicians are in being able to refer to the admirable *Jahrbuch über die Fortschritte der Mathematik*, and to

Schoute's exceedingly prompt *Revue Semestrielle*—I am sure it will be the wish of the Society that, in so far as we have ability, we should help on the good work of providing the ready subject references which those important publications do not exactly supply. The other illustration of growing international fraternity which I have in mind is one to which I should like to point with satisfaction in our own *Proceedings*. I allude to communications made to our Society by foreign mathematicians. I am, I think, right in saying that Herr Sommerfeld's paper in our volume for 1897–8 is the first lengthy one which we have printed in the German language.

The world-wide extension of the principle of collaboration, or rather perhaps co-operation, appeals to us all the more as we ourselves afford a signal example of the success of co-operation at home. The London Mathematical Society has now completed its thirty-third year of work. From small beginnings it long since raised itself to a position of controlling influence among English mathematicians, to that of a recognized exponent of British mathematical energy among those abroad. Its vitality is no longer that of youth—of this we have painful evidence when we reflect that only two original members are still on our list; that most of the leaders of its early activity are with us no more; that its first six Presidents, and not, alas! its first six only, have passed away—but its mature strength is firmly established. It is an institution representative of the state of mathematics in England as no institution has ever been before. Were I capable of the task, and were the limits of a short address adequate for the purpose, I could, I am sure, set before myself this evening few aims so worthy as that of reviewing the stages of our growth, the history of our efforts, the realization of our hopes, the establishment of our position, the work we have done, the debts we owe, the opportunities that are before us, the lessons to be learned by looking for our omissions. I must be much less ambitious. A few of the points which would suggest themselves in connexion with such a review are all that I can touch upon.

There was something almost humorously modest, as we now see it, in the beginnings of our Society; something flippant rather than sanguine in almost the final words of the opening address of its first President, De Morgan, on January 16th, 1865. "If," said he, "it should chance that we find a disposition among the members of this Society to leave the beaten track and cut out fresh paths or mend the old ones, we may make this Society exceedingly useful. But, if not—if it be our fate only to become problem makers and prob

solvers—there is no harm done; we shall but add one more association to the list of journals, colleges, &c., devoted to this object." They were the light words of a professor to his own class—a class to be helped among other ways by being taught not to be in a hurry to think too much of itself. But stimulating words, too, were addressed to that first meeting; the seed fell on good ground; and the unambitious lecture-room or local Society—for even London, not being Cambridge, was then local for mathematical purposes—rapidly expanded into a national one.

The immediate benefits to be gained from actual meetings seem naturally to have been those most thought of by the young Society. It was only hoped against fears to the contrary, as De Morgan's words just quoted sufficiently show, that great and lasting enrichment of mathematical literature by the Society's publications would follow as a sequel. But there was the opportunity, there were the men, and the initial step was the right one. Those with a common, an engrossing, interest were to come together, were to impart ideas to one another, to throw new light on each other's ideas, were to consider together points which presented themselves as to the history, the logic, the language, the widening horizon of mathematics, to thus have the ingenuity and enthusiasm which, unguided, the majority would be likely to exercise only in the production and solution of mere "ten-minute conundrums," in the same old restricted domain of elementary mathematics, directed towards more worthy problems, towards branches of mathematics where real work was to be done, away from slavery to an insular (but too much dispraised) examination-room cultivation of facility in little things towards the expansion of mathematical study and opportunities for the adaptation to widened views of mathematical education. Without co-operation only the few would realize, and they perhaps in a way not altogether practical, the openings for development; and the few would lack encouragement. The Society's meetings were to stimulate the co-operation: and they did it.

It was perhaps inevitable, but I regret it, that, as the achievements of the Society grew, the advantages to be gained from its meetings themselves by a large body of its members should diminish, that so many of us should never attend a meeting of the Society at all, that the reading, or taking as read, of memoirs of comparatively ambitious character presented for publication in our *Proceedings*, often from their nature and length difficult to state orally in lucid abstract, should come to be regarded as alone appropriate to the grave dignity

of our gatherings. There is still time on most occasions for informal discussion on matters which are not abstruse. There is still opportunity for calling attention to what is being done elsewhere in verbal explanations not offered for publication, for consultation on points which interest individual members, and of which others may be supposed to know more. There is still a larger number of members who can stimulate others in such ways as these than can hope—or rather than actually do hope—to offer elaborated original work for submission to referees. There are still those who love mathematics with an unambitious form of love, whose latent energies could be worked upon, would they come together and encourage one another, as in a young and undignified society. Let us not entirely leave this secondary, but not trifling, work, which leads on to higher things those whose tendency is to remain in easy grooves, to youthful and modest and local societies. Let us, however, be glad that there are such societies to follow our good example set in the days gone by, to profit by our old lessons, and provide us with a succession of new workers, helped to the front as our older ones were helped in this room.

I have so far dwelt on a form of work for the advancement of mathematical science, which is done by a society like our own, perhaps best in its early days of informality. But the advantages of collaboration which a society affords—which ours has afforded—are not by any means seen only in the personal intercourse of its members at meetings. Our main business has for long been the publication of real contributions to mathematical knowledge. There was no prolongation of a tentative period, in the matter of this great aim, in the early history of the Society. A glance at the first two volumes of the *Proceedings* shows how warmly and at once the young Society was supported—how the first mathematicians of the country rallied round it and offered work worthy of their names to the ears and eyes of its members. Thus started, the *Proceedings* at once assumed a prestige with which the printing in them of slipshod work was incompatible. There is, I think, a special sense of obligation to devote finishing care to a piece of work offered for publication by an established society of which all of us are sensible, a feeling that we have at stake, not only our own reputation, but to a certain extent that of the Society, which we ask to be our medium of publication, a knowledge that we are appealing to authority for support, or at any rate countenance, before giving our ideas to the world, that we invite prior criticism from experts, that we may have some le

gather from their co-operation which we will accept with thankfulness, but that before submitting our efforts for consideration we must take even unnaturally great pains to secure that in them there be no inaccuracy, inelegance, or encroachment to which a finger of disapproval might be pointed. There is a corresponding sense of responsibility in the Society's Council and referees in the discharge of their delicate duties in connexion with papers submitted. It may be that some think that a referee as a rule merely glances through a paper and expresses to the Council an opinion that it may or may not be printed as new and true. But to think this is greatly to underrate the minute care which is, as a rule, in fact devoted to doing justice at once to the author and to the Society. How few, for instance, of all those who for many years had papers in the Society's *Proceedings* had not gratefully to acknowledge some suggestion made, some help, reference, or information given, by that best and most untiring of referees, the late Professor Cayley! The whole system of publication by a Society like ours is one of friendly co-operation among authors.

It was early an aim of the Society—indeed it was a main subject of exhortation in Augustus De Morgan's first Presidential Address—to fight against narrowness of mathematical view, to use the principle of co-operation to open one another's eyes to branches of mathematics which had grown, and were growing, outside the important but restricted fields to which British attention was devoted. After twelve years of its work, the Society listened to Professor Henry Smith's most brilliant and inspiring address "On the Present State and Prospects of some Branches of Pure Mathematics," in which he enforced this aim, dealing in particular with some branches of mathematics—notably the Theory of Numbers—in which his own skill was unsurpassed, and his acquaintance with what had been done well-nigh exhaustive. He was already able triumphantly to acquit the Society of having devoted its energies to little or trivial subjects, to the problem making and solving which De Morgan had hardly been sanguine enough to fail to anticipate as the Society's main exercise; and he only half admitted that the Society had not succeeded in inducing British mathematicians to discontinue showing undue partiality towards a few great branches of growing mathematical science to the exclusion of others. Probably, had propriety admitted of his pointing to his own fine work, done largely through the Society, his half admission might have been replaced by more than half a denial. The address to which I refer I regard as mark-

ing an epoch in the Society's history, and in that of our pure mathematics. If we wish to realize to any extent what widening has been since effected in the scope of our mathematical activities, let us read once more Henry Smith's exposition of things possible and desirable. We shall, I am sure, put down the fascinating pages with a sense that many a step has been taken for which he longed, and with a feeling that he saw with a prophet's eye and preached with a prophet's power to win disciples. He pleaded not only for interest in and papers upon an extended range of mathematics, but for books —for men to penetrate deeply into subjects which had received much attention abroad, while others had been pursued at home, and to enrich the specialist literature of our own tongue with advanced and comprehensive treatises on such subjects, treatises which should occupy places of worthy companionship with those, such as Dr. Salmon's, that already existed in departments of mathematics where we were well to the front, and thus to inspire the coming mathematicians of our race with the enthusiasm in other departments, which, thanks to those existing works, already prevailed in departments where their help was with us. He appealed to members of the Society, and, through the Society, to all whom it could influence —in other words, to all English-speaking mathematicians throughout the world. Something of what he asked for remains still doing or to be done; but we reflect with satisfaction that much is done already. Have we not, for instance, more than one of the worthy books on Elliptic Functions that were hoped for? Have we not a considerable instalment of the work on the "Theory of Numbers," which he placed in the forefront of those he personally desired, given us by one we are proud to reckon among ourselves, and do we not await with keen desire the continuation promised from the same pen? Has not another gone far in the way of responding to the request for an adequate English treatment of the vast subject of differential equations? Is the theory of functions still unrepresented in the English department of our libraries, or even on the modest shelves of our junior students? Has not the "bold man" of Henry Smith's dream, rather than expectation, been found among us to grapple with Abelian transcendents? Is not a work of the last two years on the "Theory of Groups of Finite Order," embodying so much actually introduced by its author through our own Society, one which would have cheered the heart of our great President in connexion with his brief reference to the theory of substitutions? Are there not other fine works, as, for instance, one on "Universal Algebra," which

come and are coming from active members of the Society in departments of mathematics as to which no anticipation was hazarded in the address to which I have been alluding, but whose existence is another indication of the expanding energy which was so largely inspired by that address, and has been so genuinely fostered by the Society? Henry Smith did not dwell upon possibilities in applied mathematics, nor will I presume to allude to recent important books by our members in this domain, where I am so poorly qualified to say what is adequate. It would be hardly, perhaps, relevant to the subject in hand—that of the introduction to our notice of bodies of thought which had been foreign to our province—to do so. The following, or prompt or tardy, of outside leadership is not what we are most wont to think of in connexion with British applied mathematics. Our physical mathematicians have been taught to lead. It is the growth of pure mathematics in respect of which I have been congratulating the Society on the work which it has encouraged. Other influences have been at work in producing such growth among us besides those of which we think with pride in this room, for instance, the encouragement of deep special study, on the part of a select few, by the reorganization and division of the Mathematical Tripos examination, the aims and hopes connected with which were so appropriately laid before us, while yet fresh, by another of our former Presidents; but even in the inception of these hopes and aims I am fain to see that the influences which invigorated our Society did a share of the work.

Great reforms are wont to have as their temporary accompaniment minor discouragements and a passing sense of confusion. The conservative mind is for a time somewhat perplexed. Some of us do not see at once how a well established didactic system is slowly to be modified—no radical revolution is possible, even if desirable—so as best to prepare for an enlarged conception of ultimate opportunities. Some of those who may or may not join the ranks of mathematical specialists, the recipients for the time being of didactic treatment, are distracted by the apparently conflicting claims of an old order and a new. It is my object, in what time remains to me, to pay brief attention to a few considerations subsidiary to the grand one of widening investigation. Indeed, having, in accordance with precedent, to give some title to this, I fear, disjointed address, I have chosen one which only has reference to the few remarks which follow.

Unambitious work of definitely educational intention on the part of those who have received the higher enlightenment is what is

needed at such a time of transition. We believe a sound mathematical training to be of such value in mental development, even in the case of those to whom mathematics is not to afford a life's pursuit or a life's interest, that any possible numerical weakening of the band of mathematical students in our Universities and Colleges should be striven against with all our energies. We believe, and lay stress here on our belief, in the fact that life-long application to mathematical investigation, on the part of those who prove to be qualified for it, is a noble use of the human mind, a devotion to the pursuit of truth in a field where the human intellect is on sure ground. The two claims on our fostering care are not identical, but to let either be hampered is also to discourage the other. While we all desire that the domain of mathematical exploration among us be widened as much as possible, we equally, I hope, desire that the number of explorers be not few. The fascination exercised by mathematics on a large class of minds is a great reality. We wish to see that fascination grow, and not be checked. We cannot shut our eyes to the fact, but must endeavour without discouragement to utilize it, that it is the gymnastic of mathematics which has most captivated the intelligence among us British. It is a manifestation of the leaning of our race, and not merely a result of our educational system, that so many minds among us have too long been satisfied by attention to minute elegancies, to small isolated but complete problems. They will long, I hope, continue to be much occupied, though not satisfied, by such matters. I am not among the despisers of the British problem; not one of those who regard time such as I spent in days gone by, as a student, in concentration on sifting small matters to the bottom as time ill spent, or the delight experienced when an ingeniously proposed difficulty was conquered, or the full beauty of a minor truth realized, as idle gratification; not one of those who regret as waste labour the care since devoted to the construction of many an examination paper full of special minor applications of truths one can never know too thoroughly or too diligently enforce. What is it which above all things makes it a less irksome and more satisfying task for an English reader, however proficient a linguist, to acquire new mathematical knowledge from a book by an English author than from a foreign one? It is not only that the language is his own, that the expression is of the kind to which he is accustomed. It certainly is not that the arrangement is often more lucid and systematic. It is mainly that his own intelligence is so often called into play in detail, and the effort to fo[illegible]

long succession of connected arguments relieved, by illustrative examples introduced to fix his attention, to make him thoroughly appreciate points reached, to exercise him in anticipating further developments for himself, to give his practical form of business-like sense notions of definite, though it may be in themselves trifling, applications to which he is already put in position to attain. His mastery of a whole theory is thus perhaps delayed; but his growing knowledge is made enormously more thorough, more appreciative. He grasps the full force of arguments used, learns how to refine methods of proof, to exhibit facts in their relations to one another, and in new bearings. The doing little things for himself leads him on to greater. Thus problem solving, and its sequel problem making, I regard as in a high degree beneficial. Nor do I admit that it is directly unproductive of beautiful and important theory. The examples in our great dynamical text-books, for instance, the examination papers from which so many of them are taken, and in which so many which are as good remain buried, contain in many cases sketches of masses of theory which might well repay detailed exhibition and a more self-assertive publicity. They would probably often have this publicity elsewhere; and I am very glad that the old English notion that the appearance of a piece of original mathematics in a University or College examination paper amounts to publication, a notion which has kept in the background valuable discoveries for generations, is showing a tendency to disappear.

Even those, and they are, I believe, many, who do not agree with me in thinking that to throw contempt on problem making and solving is to discourage a laudable passion for perfection in detail, and to hamper the action of a real incentive to higher effort, must allow that the prevailing spirit which they regard as evil is one which cannot readily be exorcised, but must be reckoned with in making efforts to guide mathematical energy aright, and give it encouragement. Cannot the region of activity of the slave of minor interests, if we must call him so, or the searcher for yet hidden wealth where riches have been already found, if we may more kindly regard him, be extended, so that his interests, though subordinate, may be in what may repay servitude, or so that he may mine where the rich ore is far from exhausted? We want to lead the tyro where there is new work to be done, to guide him sufficiently far along toilsome routes which pioneer mathematicians have traversed, that he may find new fields from which to extract the fruitfulness, and, it may be, new paths along which to pass in further discovery. Make

it easy to go a little way in unfamiliar mathematical studies, and point out clearly something of the aims of those studies, some of the recorded achievements of those who have gone far in them, and the inducement will be strong for some who would retreat to advance, for many who might be content with culling flowers in the same old garden to adventure where they may expect a newer and richer growth. There is unlimited capacity among us to settle and develop resources, to make the very best use of opportunities at hand. There are, on the other hand, few of our race who do not recoil at the thought of a long course of sustained effort in following step by step the journeys of another, if there be no assurance or dream that lands of promise border and lie beyond his path.

What we above all things want is, I believe, a varied production of modernized didactic text-books. I have congratulated the Society on the work of recent years, largely inspired by itself, in the production of ambitious treatises calculated to exhibit to the inner circle of accomplished mathematicians a fuller knowledge of recent mathematical advances, calculated to induce those who are already real researchers to research nearer the present confines of known mathematical truth, to give larger views to those who are to lead on coming mathematicians. The next thing is for those whose views are enlarged to do their duty as leaders by popularizing sound doctrine for the benefit of the rank and file, by endeavouring to secure that the elementary teaching of mathematics be as captivating as ever, but so conveyed that thought be encouraged, that attention to logical soundness in fundamentals be enforced as essential in real mathematics, and by providing lucid and suggestive introductory works on higher subjects, suited to be at once studied by those who have acquired the gift of accurate thought and the possession of elementary knowledge.

The reform of teaching in fundamentals is, I am glad to think, beginning to receive adequate attention. A very recent book, for instance, on elementary algebra, marks a very strenuous effort in that direction. There are distinct signs that the era of elementary and quasi-elementary works of the neat but superficial order—which try to hide difficulties rather than to elucidate them or present them as matters for thought, which aim at presenting what may pass for demonstrations in the briefest form for writing out without waiting to enforce lessons of accuracy, and opportunities for intelligent consideration of principle—is passing away. I pause to make the trite remark that the opportunity for closing, the responsibility for

continuing, this era rests largely with the conductors of examinations —and examiners are happily often taken from the ranks of those qualified to lead aright. I have already made it clear—perhaps too clear—that I don't want examiners to discontinue setting problems, to test realization of principle or even ingenuity and technical skill. But there is a kind of examination question, abounding, for instance, in analytical trigonometry and in the early differential calculus, which is of most pernicious influence, a kind of request to which an examinee can only give a sound logical answer, if at all, with great expenditure of time, and to which a conventional one which practically shirks the difficulty can be written out in a few minutes. While such questions are set, elementary text-books, which must continue to be largely written by practical teachers who have in view the battle against examiners, will continue to present the conventional, imperfect, and even misleading, though it be.

But the books which I am intending to express a desire to see multiplied, that an enlargement of the range of mathematical study among us may become general and welcome, are not the most elementary ones, not even such much needed ones as might deal with fundamental principles of infinitesimal calculus with the full attention which has been bestowed abroad. We want a greater number of books, and in particular unassuming partial and introductory books, on advanced, on modern, subjects. The demand for a leader in satisfying our need for extended works has, we have seen, in many subjects been satisfied. The work done in satisfying that need is all the more clearly patent when the leader ceases to be a sole English authority on his subject. Take the subject of elliptic functions, which it happily seems already out of date to speak of as one only recently represented by books in English. The date of the appearance of Professor Cayley's work, which removed the stigma of neglect in this one field from English authorship, exactly coincides with that of Henry Smith's appeal which I dwelt on just now. The appealing address announced its completion as the beginning of the supply of its list of desiderata. Its publication worthily met a crying need; but it invited followers, far from blocking the way by its authority. The later book by Professor Greenhill on the subject was all the more welcomed because the way had been prepared. And both the one and the other enforced the desirability of a short and professedly didactic introduction, such as that which we afterwards owed to Mr. A. C. Dixon. Even now we know that there is room for more, that in particular an introduction to the subject from the theory of

functions point of view, one almost exclusively adopted in recent foreign works upon it, would have the greatest interest and instructiveness. Take, again, the vast aggregate of modern mathematics included in the comprehensive title Theory of Functions. Who can regret that it was introduced to the English reader in two practically simultaneous works, one by Dr. Forsyth and the other by Messrs. Harkness and Morley, instead of by one man only? Who is not benefited by taking the one with the other? Who, seeing the enormous range of investigation referred to in the two works, does not see opportunities for other authors to treat with advantage some of them, not only by pushing on to higher developments, as Mr. Baker has since done in the matter of Abel's theorem, but in more modest ways too? The exceeding clearness of Professor Forsyth's elucidation of first conceptions may not easily be surpassed; but, in referring a moment ago to one desirability in connexion with the elliptic functions in particular, I have instanced a case of opportunity for secondary and didactic treatises.

There is a wide realm to be made more fully appreciated among British mathematicians in the *continuierliche Transformationsgruppen* —I could wish we had a better translation of the words than "*continuous* groups of transformations"—whose theory is the fruit of the genius of our foreign honorary member Professor Sophus Lie. It is a realm as yet hardly touched, I believe, by text-books in English on this side of the Atlantic, though Page's *Differential Equations* has begun the work on the other. It cannot, I trust, be much longer before didactic helps to the study of different parts, and applications, of this prodigiously far-reaching theory are given us from among ourselves. Direct participation in its general investigations has been slow in finding a place in the labours of our researchers; but, thanks mainly to Professor Burnside and Mr. J. E. Campbell, it is now taking a prominent place in our *Proceedings*. Under the auspices of our younger sister the American Mathematical Society, whose members perhaps are more directly under the influence of the German Universities than our own, very much is being done.

To another consideration in connexion with Lie's magnificent enrichment of analysis I should like to allude. A secondary reason for welcoming the signs that it will soon be given due prominence among us is that it affords so much opportunity for collateral examination into beautiful detail—so many an opening for the useful application of the passion for thorough examination of incidentals which I have admitted, and even boasted, to be ineradicable among us. When a

great theory is rapidly poured forth on the world, all that is to be said at each stage cannot be included. The main lines of advance occupy the pioneer's attention. He is pushing ahead into a new continent. He surveys the immediate left and the immediate right as he passes, but it is onward that he strains every nerve to proceed. Fields rich in detailed interest are left behind for those who follow to explore and permanently utilize. Many will remember the enthusiasm with which Sylvester some thirteen years ago announced his conception of "Reciprocants," a body of differential invariants not for a group but for a mere interchange of variables. A considerable number of us caught the fire, and participated in consequential investigations as to orthogonal linear and projective groups, groups in whose transformations interchanges of variables occur as particular cases, and whose differential invariants are consequently classes of reciprocants, and of the analogues of reciprocants when more variables than two are considered. The investigations in question were long subsequent to Lie's consideration of the groups in question as leading cases of a general conception, but were carried out in most instructive detail. They were secondary investigations—the fact, alas! was very inadequately grasped—but they were not trivial. I do not like to think of any knowledge gained in the sure domain of mathematical truth as trivial. Even in this well traversed subtheory, I should be loath to admit that the researcher of keen insight cannot yet be repaid. For instance, Sylvester and others devoted much pains to the construction of a syzygetic theory of pure reciprocants, *i.e.*, differential invariants of the "special" sub-group of the linear group. That theory, to be placed in position to advance from an absolute basis of certainty, asks for the demonstration of a certain theorem, akin to one of Cayley's, which long awaited its first proof by Sylvester, as to the number of seminvariants of given type possessed by a binary form. The use of a fundamental differential operator of infinitesimal transformation indicates that the number of pure reciprocants of given characteristics, called their weight, degree, and extent, is not less than the excess of one number of partitions over another, and observation as far as carried suggests that the number is equal to that excess when it is positive and zero in other cases. So far as I know, the accuracy of the suggested law has only been established when weight is not less than degree, a state of things under which there is a one to one correspondence of pure reciprocants and seminvariants. But cannot the secret be unravelled? Cannot the discriminating law which separates cases of positive or

non-negative excess from others be in some way extracted from the theory of differential operators? The quest may seem one which bears only on a minor theory of a purely formal character; but was not the whole consideration of asyzygetic concomitants of a binary quantic once but that of a minor theory in the same sense? May not a flood of light be thrown on a region of the combinatory analysis when the whole truth, of which what at present appears is an incomplete and perhaps an inexact indication, is manifested?

I turn to another need whose satisfaction would go far towards the more rapid introduction to our attention of advances made, and towards the stimulation of widened interests. We suffer from a deficiency of historical and bibliographical literature. We have, I know, one journal, *Nature*, which supplies us with notes on subjects of mathematical interest, and with most discriminating reviews of new mathematical works, both home and foreign, as they appear. The initials G. B. M., and others, which we see appended to the reviews are those of friends whose services in this respect to our mathematicians, actual and coming, deserve from us the warmest recognition. We desire, however, something more than there can be space for in a journal which represents mathematics only as one of many sciences. We desire lengthened expositions of the scope of new theories, and of older ones in which too few of us have learned to take an interest; analyses of specially important memoirs which are only likely to meet the eyes of few of us till it is shown that they will repay a special effort to consult, or of sections of peculiarly instructive foreign books which few individuals would be likely to buy or libraries to acquire till something of the wealth within them was clearly indicated; explanations of the bearing of modern achievement on the remodelling, the illumination, or the unification of what has been long before us. Incalculable service would be done to our breadth of view by a growth among us of such literature; and I may add that in the production of it there might be a utilization for the common benefit of a form of mathematical strength which has little opportunity for exercise in our behalf—the strength of the wide reader who grasps and can explain all, but who despises a minor investigation, and never embarks on a great one because he feels, as some do even in England, that there must be no hiatus in his knowledge of all that has been done in advance before he can presume to begin extending. The supply of the need in our language for this awakening form of literature is now coming to us from the far side of the Atlantic. A great work is being done in the

Bulletin of the American Mathematical Society, an institution which has profited from our own Society's example, and also strikes out new lines for itself. Can we not do more on this side of the water too? I notice with great satisfaction Dr. Lovett's paper on "The Theory of Perturbations and Lie's Theory of Contact Transformations," in Nos. 117, 118 of the *Quarterly Journal*, and rejoice if at least one of our English mathematical journals is open to communications of this nature. As to ourselves, as a society, we have, I fear, not the space or the funds which would justify us in inviting such contributions for our own publication. I have earlier, however, expressed a feeling that at our meetings we should have much to gain from oral instruction of the like character.

I crave pardon if some of the remarks now made to this Society, whose aim is the highest possible in mathematics, would have seemed more in place before a society whose direct concern is chiefly with the interests of mathematical education. I am convinced, however, that the interests of research, which are our main care, have for a part the interests of the training of future researchers. Those interests demand that such of us as have been helped on should use their influence to help on increasing numbers, to secure that what was serviceable to ourselves in making us workers at the higher mathematics may be even more serviceable to others, and that any compensating disadvantages or difficulties which stood in our way may disappear.

In conclusion, let me revert for a moment to a topic mentioned just now—one of those on which I must have dwelt had I been bold enough to follow my first inclination and make a sketch of the history of the Society my theme to-night—that of the debts we owe. There are two great debts, closely associated, which it is a time to specially acknowledge. For well-nigh thirty years all acts of the Society bore the signatures M. Jenkins, R. Tucker, Secretaries. Our Secretaries are Honorary Secretaries. Their work for us has been a labour of love—"I love y^e Society" one of them wrote to me years ago, using words which might have come with equally obvious truth also from the other—and no light labour. Three years ago Mr. Jenkins found it necessary, to the Society's keen regret, to resign the onerous duties he had so generously and so efficiently discharged since the infancy of the Society. He was prevailed upon still to help us as Vice-President, and on the Council; but to-night we have had, at his own strong wish expressed in an affectionate message of farewell, to omit his name from the Council elected. We thank him

with deep sincerity for all his services rendered. His extreme modesty has always made him prefer to figure as if his work for us were light and his mathematical distinction inconsiderable. But we have known far better, in the one matter and the other. Mr. Tucker remains to us, to begin his thirty-third year on the Council and his thirty-second as Secretary. Nearly all the papers which the Society has ever received have caused him correspondence. Sixteen Presidents have, like myself, found their office free from anxiety because of his and his colleague's assiduity. There has been no limit to the burdens he would willingly take upon himself in his absolutely unselfish devotion to the interests of the Society. Such a use of what might have been the leisure of half a life-time has put mathematical science under an obligation for which no gratitude would be excessive. May he long be good enough, and have the health and strength, to add to this load of obligation!

On the Functions Y and Z which satisfy the Identity

$$4(x^p-1)/(x-1) = Y^2 \pm pZ^2,$$

where p is a Prime of the Form $4k\pm 1$. *By* L. J. ROGERS. Received November 7th, 1898. Communicated November 10th, 1898.

§ 1.

In Prof. Mathews's *Theory of Numbers*, pp. 215–219, a very full account is given of the resolution of $4X, = 4(x^p-1)/(x-1)$ into the form $Y^2-e_1pZ^2$, where Y and Z are integral functions and

$$e_1 = (-1)^{\frac{1}{2}(p-1)},$$

according to his notation on pp. 216 and 217.

On these latter pages a method is given for calculating successively the several coefficients which occur in Y and Z, with a remark that it would be desirable to discover a method of writing down their general values, without having to calculate the preceding coefficients. The object of this present paper is to show how this may be done,

and to further point out certain properties of the functions Y and Z, whereby the coefficients may be more easily deduced.

It is well known that

$$\left.\begin{aligned} Y-\sqrt{e_1 p}\,Z &= 2\,(x-r^{g^2})(x-r^{g^4})\,\ldots \\ Y+\sqrt{e_1 p}\,Z &= 2\,(x-r^{g})\,(x-r^{g^3})\,\ldots \end{aligned}\right\}, \tag{1}$$

where r is any complex p^{th} root of unity, and g is a primitive root of p.

Denoting by $\dot{Y}$ and $\dot{Z}$ the functions $\dfrac{dY}{dx}$ and $\dfrac{dZ}{dx}$, we have

$$\frac{\dot{Y}-\sqrt{e_1 p}\,\dot{Z}}{Y-\sqrt{e_1 p}\,Z}=\frac{1}{x-r^{g^2}}+\frac{1}{x-r^{g^4}}+\frac{1}{x-r^{g^6}}+\ldots,$$

$$\frac{\dot{Y}+\sqrt{e_1 p}\,\dot{Z}}{Y+\sqrt{e_1 p}\,Z}=\frac{1}{x-r^{g}}+\frac{1}{x-r^{g^3}}+\frac{1}{x-r^{g^5}}+\ldots,$$

whence, by subtraction,

$$2\sqrt{e_1 p}\,\frac{\dot{Y}Z-Y\dot{Z}}{4X}=\sum_{h=1}^{p-1}\frac{(h/p)}{x-r^h}$$

[where (h/p) is the usual Legendrian symbol]

$$=\frac{1}{x}\Sigma\,(h/p)+\frac{1}{x^2}\Sigma\,(h/p)\,r^h+\frac{1}{x^3}\Sigma\,(h/p)\,r^{2h}+\ldots.$$

Now $\quad \Sigma\,(h/p)\,r^h=\sqrt{e_1 p}\;$ and $\;\Sigma\,(h/p)\,r^{mh}=(m/p)\sqrt{e_1 p},$

or, using the notation before referred to,

$$\frac{\dot{Y}Z-Y\dot{Z}}{2X}=\frac{1}{x^2}+\frac{e_2}{x^3}+\frac{e_3}{x^4}+\ldots \text{ to infinity},$$

where $e_m=(m/p)$, when $m>1$ and prime to p, while $e_{mp}=0$. Thus

$$\frac{\dot{Y}Z-Y\dot{Z}}{2X}=\frac{1}{x^p-1}\,(x^{p-2}+e_2x^{p-3}+\ldots),$$

i.e.,
$$x\,(x-1)(\dot{Y}Z-Y\dot{Z})=2\,(x^{p-1}+e_2x^{p-2}+\ldots\, e_{p-1}x). \tag{2}$$

It is, however, somewhat more convenient to reverse the polynomials Y and Z so as to consider the successive coefficients of ascending powers of x, and, in the infinite series which follow, to suppose $x<1$. These reversed series will be denoted by y and z, with

the supposition that their leading terms are positive. By the known reciprocal properties of Y and Z, we have then

$$y = e_1 Y \quad \text{and} \quad z = Z,$$

while $$x^{p-1}+e_2x^{p-2}+\ldots = e_1(x+e_2x^2+e_3x^3+\ldots e_{p-1}x^{p-1}).$$

Hence (2) becomes

$$x(x-1)(\dot{y}z-y\dot{z}) = 2(x+e_2x^2+e_3x^3+\ldots). \tag{3}$$

Now let

$$\left.\begin{aligned} y &= 2\sqrt{X}\cos\theta \\ \sqrt{-e_1 p}\, z &= 2\sqrt{X}\sin\theta \end{aligned}\right\}; \tag{4}$$

so that (3) becomes

$$-4Xx(x-1)\,\dot{\theta} = 2\sqrt{-e_1p}\,(x+e_2x^2+\ldots),$$

$$2\dot{\theta} = \sqrt{-e_1p}\,(1+e_2x+\ldots)/(1-x^p)$$

$$= \sqrt{-e_1p}\,(1+e_2x+e_3x^2+\ldots \text{ to infinity}),$$

and $$2\theta = \sqrt{-e_1p}\,(x+\tfrac{1}{2}e_2x^2+\tfrac{1}{3}e_3x^3+\ldots), \tag{5}$$

no constant being added, since, when $x=0$, $z=0$, $\theta=0$. Substituting in (4), we have

$$y = 2(1-x^p)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}\left\{\begin{aligned} &1+\tfrac{1}{4}e_1p\,(x+\tfrac{1}{2}e_2x^2+\ldots)^2 \\ &+\frac{1}{2^4}\,\frac{1}{24}p^2\,(x+\tfrac{1}{2}e_2x^2+\ldots)^4 \\ &+\ldots \end{aligned}\right\},$$

$$z = (1-x^p)^{\frac{1}{2}}(1-x^{-\frac{1}{2}})\left\{\begin{aligned} &(x+\tfrac{1}{2}e_2x^2+\ldots) \\ &+\tfrac{1}{4}\cdot\tfrac{1}{6}e_1p\,(x+\tfrac{1}{2}e_2x^2+\ldots)^3 \\ &+\frac{1}{2^4}\cdot\frac{1}{5!}p^2\,(x+\tfrac{1}{2}e_2x^2+\ldots)^5 \\ &+\ldots \end{aligned}\right\}.$$

From these equations any coefficient in y or z can be calculated without the knowledge of the preceding coefficients, and the results are the same in form as those obtained in Prof. Mathews's treatise, p. 217, These formulæ are, however, extremely intricate, and, considering that the coefficients reduce in all cases to integers, are remarkably fractional. Their ultimate simple form no doubt depends upon some inherent property relating to the order of the quadratic residues of p, but it is not easy to see how such a reduction in general can be effected.

It may be interesting, then, to point out two other methods for obtaining the required coefficients.

§ 2.

Let $$U \equiv 1+2\,(x+x^{\alpha}+x^{\beta}+\dots),$$

where 1, α, β are the quadratic residues of p which lie between 0 and p, so that the right-hand side of § 1 (3), is $2\,(U-X)$.

Now, when $$x = r^{g^2}, \quad U = \sqrt{e_1 p},$$

but, when $$x = r^{g}, \quad U = -\sqrt{e_1 p}.$$

Again, when $$x = r^{g^2}, \quad Y = \sqrt{e_1 p}\,Z,$$

but, when $$x = r^{g}, \quad Y = -\sqrt{e_1 p}\,Z.$$

Hence $UY-e_1pZ$ and $UZ-Y$ are zero for all values of r, and therefore contain X as a factor.

Moreover, both functions $\equiv 0 \pmod 2$, since

$$U \equiv 1 \quad \text{and} \quad Y \equiv e_1 pZ \equiv Z.$$

Let us write then $$\left.\begin{aligned} UY-e_1pZ &= 2MX \\ UZ-\;Y &= 2NX \end{aligned}\right\};\tag{1}$$

or, what is the same thing,

$$\left.\begin{aligned} 2U &= MY-e_1pNZ \\ 2 &= MZ-NY \end{aligned}\right\}.\tag{2}$$

Similarly, if $V = 2X-U$, *i.e.*, if $\frac{1}{2}(V-1)$ be the sum of powers of x whose indices are $<p$, and non-residues of p, then

$$\left.\begin{aligned} &VY+e_1pZ \text{ may be written } 2XM' \\ \text{and}\quad &VZ-Y \qquad\quad \text{,,} \qquad\qquad 2XN' \end{aligned}\right\},\tag{3}$$

so that $$M+M' = Y,$$
$$N+N' = Z,$$
$$2V = M'Y-e_1pN'Z.$$

Now, if $e_1 = -1$, then $p-1$ is not a residue, so that U is of lower order than X. Consequently M is of lower order than Y, and N of lower order than Z. On the other hand, if $e_1 = 1$, then $p-1$ is a residue, and V is of lower order than X. In this case, M', N' are respectively of lower order than Y, Z.

In all cases therefore it is possible to find numerically integral polynomials in x, say μ and ν, such that

$$\mu Z-\nu Y = 2.$$

Moreover, if Y and Z are known, M and N may be very easily deduced by differentiation, for

$$x(x-1)(\dot{Y}Z-Y\dot{Z}) = 2e_1(U-X), \quad \text{by § 1, (2)},$$

while $$(x-1)(Y\dot{Y}-e_1pZ\dot{Z}) = 2p-2xX+2p(x-1)X,$$

by differentiating the identity

$$Y^2-e_1pZ^2 = 4X.$$

Hence $$4Xx(x-1)\dot{Y} = 2pY-2xXY+2p(x-1)XY-2UZp+2XZp,$$

$$4Xx(x-1)\dot{Z} = 2pZ-2xXZ+2p(x-1)XZ-2e_1Y(U-X).$$

Substituting for U, from (1), we have

$$2e_1M = p(x-1)Z-xZ+e_1Y-2x(x-1)\frac{dZ}{dx}$$

$$= p(x-1)Z+xZ+e_1Y-2x\frac{d}{dx}(x-1)Z,$$

which is a form more adapted for the numerical calculation of M. Similarly,

$$2pN = p(x-1)Y+xY+pZ-2x\frac{d}{dx}(x-1)Y.$$

When $e_1 = 1$, it is better to use the equivalent equations

$$2e_1M' = -p(x-1)Z-xZ+e_1Y+2x\frac{d}{dx}(x-1)Z,$$

$$2pN' = -p(x-1)Y-xY+pZ+2x\frac{d}{dx}(x-1)Y.$$

§ 3.

We may combine the results of the previous section by writing

$$2W = (U+V)-e_1(U+V),$$

$$2\mu = (M+M')-e_1(M-M'),$$

$$2\nu = (N+N')-e_1(N-N'),$$

so that $W, \mu, \nu = U, M, N$ or V, M', N' according as $e_1 = -1$ or $+1$.

With this notation, § 2, (2) and (4) give us

$$\left.\begin{aligned} 2W &= \mu Y-e_1p\nu Z \\ 2 &= \mu Z-\nu Y \end{aligned}\right\}, \qquad (1)$$

where μ, ν are integral and of lower order than Y, Z respectively.

Now let Z/Y be converted into a chain-fraction of the form

$$\frac{1}{2x+b_1+}\,\frac{n_2}{m_2\phi_2(x)+}\,\frac{n_3}{m_3\phi_3(x)+}\cdots\frac{n_h}{m_h\phi_h(x)}, \tag{2}$$

where the m's and n's are positive or negative integers, and the leading coefficients in $\phi_2(x)$, $\phi_3(x)$, &c., are all unity.

If $\frac{P_1}{Q_1}$, $\frac{P_2}{Q_2}$, ..., $\frac{P_h}{Q_h}$ be the successive convergents, derived without numerical reduction from the equations

$$P_r = m_r\phi_r(x)\,P_{r-1}+n_rP_{r-2},$$

$$Q_r = m_r\phi_r(x)\,Q_{r-1}+n_rQ_{r-2},$$

then, evidently,
$$\left.\begin{aligned} Q_h &= m_2m_3\dots m_hY \\ P_h &= m_2m_3\dots m_hZ \end{aligned}\right\}.$$

Since, moreover, the fraction

$$\frac{n_{h-1}}{m_{h-1}\phi_{h-1}(x)+}\,\frac{n_h}{m_h\phi_h(x)} = \frac{n_{h-1}m_h}{m_{h-1}m_h\phi_{h-1}(x)+}\,\frac{n_h}{\phi_h(x)},$$

it is obvious that we can with equal generality suppose that $m_h = 1$, so that

$$\left.\begin{aligned} Q_h &= m_2m_3\dots m_{h-1}Y \\ P_h &= m_2m_3\dots m_{h-1}Z \\ \text{while}\qquad Q_{h-1} &= m_2m_3\dots m_{h-1}\mu \\ P_{h-1} &= m_2m_3\dots m_{h-1}\nu \end{aligned}\right\}. \tag{3}$$

Again, by the laws of chain fractions, we have

$$\frac{\mu}{Y} = \frac{Q_{h-1}}{Q_h} = \frac{1}{\phi_h(x)+}\,\frac{n_h}{m_{h-1}\phi_{h-1}(x)+}\cdots\frac{n_2}{2x+b_1},$$

$$\frac{\nu}{Z} = \frac{P_{h-1}}{P_h} = \frac{1}{\phi_h(x)+}\,\frac{n_h}{m_{h-1}\phi_{h-1}(x)+}\cdots\frac{n_3}{m_2\phi_2(x)}.$$

Now consider the fraction

$$F \equiv \frac{1}{\phi_h(x)+}\,\frac{n_h}{m_{h-1}\phi_{h-1}(x)+}\cdots\frac{n_2}{2x+b_1-}\,\frac{e_1p}{2x+b_1+}\,\frac{n_2}{m_2\phi_2(x)+}\cdots\frac{1}{\phi_h(x)},$$

which differs from $\frac{\mu}{Y}$, by writing $2x+b_1-\frac{e_1pZ}{Y}$ instead of $2x+b_1$, in

deriving $\frac{\mu}{Y}$ from its two preceding convergents. Thus

$$F = \frac{\mu - \nu \dfrac{e_1 pZ}{Y}}{Y - Z\dfrac{e_1 pZ}{Y}} = \frac{\mu Y - e_1 p\nu Z}{Y^2 - e_1 pZ^2} = \frac{W}{2X}.$$

Hence, if $\frac{W}{2X}$ be chain-fractionized, the converging denominators of degree $\frac{1}{2}(p-3)$ and $\frac{1}{2}(p-1)$ will be numerically proportional to Z and Y; while the corresponding numerators are multiples of μ and ν.

As an example we may put

$$p = 23,$$

$$W = U = 2(x^{18}+x^{16}+x^{13}+x^{12}+x^9+x^8+x^6+x^4+x^3+x^2+x)+1,$$

$$\frac{U}{2X} = \frac{1}{x^4+x^3+1-}\,\frac{4}{2x^2+3-}\,\frac{2}{2x-1+}\,\frac{1}{2x+7+}\,\frac{32}{x-5-}\,\frac{1}{2x-1-}$$
$$\frac{8}{2x-1+}\,\frac{23}{2x-1+}\,\dots,$$

whence $\quad \mu = 2x^7-x^6-4x^5-4x^4-5x^3+2x^2+10x-1,$

and $\quad \nu = x^6-x^3-2x^2+1,$

while the values of Y and Z agree with those given in Prof. Mathews's treatise, p. 218.

It is easy to see that, although we have a means of calculating Y and Z which is very simple in theory, yet in practice it involves great labour, and work of such a kind as to give chances of numerical errors.

§ 4.

The most practical way of determining the coefficients of Y and Z is derived from the equation (3) in § 1, by eliminating $\dot{z}$.

We have, namely,

$$\begin{aligned} e_1 pz\,(x+e_2x^2+\dots) &= \tfrac{1}{2}e_1 px\,(x-1)(\dot{y}z^2 - yz\dot{z}) \\ &= \tfrac{1}{2}x\,(x-1)\,\dot{y}\,(Y^2-4X) - \tfrac{1}{2}x\,(x-1)\,y\,(y\dot{y}-2\dot{X}) \\ &= 2x\,(1-x^p)\,\dot{y} - x\,(1-x)\,\dot{X}\,y. \end{aligned}$$

But $\quad (1-x)\,\dot{X} - X = -px^{p-1},$

so that $2x\,(1-x)(1-x^p)\,\dot{y} - x\,(1-x^p)\,y + px^p\,(1-x)\,y$

$$= e_1 pz\,(1-x)(x+e_2x^2+e_3x^3+\dots). \qquad (1)$$

Now, as we are only concerned with coefficients in y and z as far as $x^{\frac{1}{2}(p-2+e_1)}$, we may neglect powers higher than this, and replace (1) by

$$2x(1-x)\,\dot{y}-xy \equiv e_1 pz\,(1-x)(x+e_2x^2+\dots) \quad (\text{mod } x^p-1).$$

Similarly, $\quad 2x(1-x)\,\dot{z}-xy \equiv y\,(1-x)(x+e_2x^2+\dots.$

If, then,

$$y = 2+x+c_2x^2+c_3x^3+\dots,$$
$$z = x+f_2x^2+f_3x^3+\dots,$$

we get

$$x^2(4c_2-3)+x^3(6c_3-5c_2)+x^4(8c_4-7c_3)+\dots$$
$$= e_1 p\,(x+f_2x^2+f_3x^3+\dots)\,\{x+(e_2-1)\,x^2+(e_3-e_2)\,x^3+\dots\}, \quad (2)$$

and

$$2x+x^2(4f_2-3)+x^3(6f_3-5f_2)+\dots$$
$$= (2+x+c_2x^2+c_3x^3+\dots)\{x+(e_2-1)\,x^2+(e_3-c_2)\,x^3+\dots\}. \quad (3)$$

From these two equations we can very easily deduce $c_2, f_2, c_3, f_3, \dots$ successively, the advantages of the method resting on the integral form of all the terms on the right-hand sides, while the necessarily integral forms ultimately obtained for the c's and f's avoid the possibility of numerical errors.

For instance, if $p = 67$, $e_1 = -1$, and the quadratic residues as far as 18 numerically are $1, -2, -3, 4, -5, 6, -7, -8, 9, 10, -11, -12, -13, 14, 15, 16, 17, -18, \dots$, so that

$$x+(e_2-1)\,x^2+\dots$$
$$= x-2x^2+4x^4-2x^5+2x^6-2x^7+2x^9-2x^{11}+2x^{14}-2x^{18}+\dots,$$

whence, deducing $c_2 = -16$, $f_2 = 0$, $c_3 = 9$, $f_3 = -3$, &c., we get

$$Y = 2x^{33}+x^{32}-16x^{31}+9x^{30}+33x^{29}-44x^{28}-18x^{27}+79x^{26}-39x^{25}-48x^{24}$$
$$+75x^{23}-35x^{22}-14x^{21}+69x^{20}-89x^{19}+10x^{18}+106x^{17}-$$

with the reciprocal terms $-106x^{16}-10x^{15}-\dots$;

$$Z = x^{32}-3x^{30}+3x^{29}+4x^{28}-8x^{27}+x^{26}+9x^{25}-8x^{24}-x^{23}+7x^{22}-8x^{21}$$
$$+5x^{20}+5x^{19}-14x^{18}+8x^{17}+$$

with the reciprocal terms $8x^{16}-14x^{15}+\dots$.

Thursday, December 8th, 1898.

Lt.-Col. CUNNINGHAM, R.E., Vice-President, in the Chair.

Fifteen members present.

The minutes of the last meeting were read and confirmed.

The following gentlemen were elected members :—Robert Judson Aley, A.B., A.M., Ph.D., Professor of Mathematics, Indiana University, Bloomington, Indiana, U.S.A.; Ernest William Barnes, B.A., Fellow of Trinity College, Cambridge, R.M. Academy, Woolwich; John Hilton Grace, B.A., Fellow of St. Peter's College, Cambridge; Frank Morley, Sc.D. Cambridge, Professor of Mathematics in Haverford College, Pennsylvania, U.S.A.; Charles Almeric Rumsey, B.A., formerly Scholar of Trinity College, Cambridge; John Thomas Walley, M.A., Fellow of Jesus College, Cambridge, Assistant Professor of Mathematics, Aberystwyth.

A letter from the Auditor, Mr. Gallop, announcing that he had duly audited the accounts of the Society for the Session 1897–98, was read. On the motion of Prof. Hudson, seconded by Mr. Berry, a vote of thanks to the Auditor was carried unanimously.

Major MacMahon communicated a discovery he had recently made in the Theory of Compound Partitions.

Mr. J. E. Campbell read a paper "On Simultaneous Partial Differential Equations."

Messrs. Hammond and Berry made remarks on the communications.

The following papers were communicated in abstract :—

On Hyperplane Coordinates: Mr. W. H. Young.

Two Problems of Wave Propagation at the Surface of an Elastic Solid, and The Influence of Gravity on Waves in an Elastic Solid, with especial reference t the Earth: Mr. T. J. Bromwich.

On a Theorem in Determinants allied to Laplace's: Prof. W. H. Metzler.

Lt.-Col. Allan Cunningham, R.E. (Mr. Tucker, *pro tem.*, in the Chair), drew attention to the three following exceptionally high numbers :—

$$N_1,\, N_2 = \left[2^{213}\,(2^{209} \pm 1)^3 \mp (2^{211} \pm 1)^3\right] = (2^{210} \pm 1)(2^{210} \mp 1)^3,$$

$$N_3 = \left[\,\{(2^{105}+1)^4 - 2^{108}(3\,.\,2^{104}+1)\}^2 + \{(2^{105}+1)^4 - 2^{212}(2^{106}+3)\}^2\right]$$

$$= 2\,(2^{210}+1)^4.$$

The *complete factorization* of the numbers $(2^{210} \pm 1)$ being known (see Lucas's memoir *Sur la Série récurrente de Fermat*, Rome, 1879, pp. 9, 10), the three large numbers (N) are also *completely factorizable into their prime factors*. The two N_1, N_2 are of order 2^{840}, and therefore contain 253 figures; whilst N_3 is of order 2^{841}, and therefore contains 254 figures. The largest number hitherto *completely factorized* into its prime factors (so far as known to the author) is $(2^{210}+1)$, which contains 64 figures.

The following presents were made to the Library :—

"Mathematical Questions with their Solutions from the 'Educational Times,'" Vol. LXIX., 8vo; London, 1898.

Bashforth, F.—"Replica di Krupp alla protesta del Signor Bashforth," translated by F. Bashforth, B.D., 8vo; Cambridge, 1898.

"Proceedings of the Royal Society," Vol. LXIV., No. 404.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. V., No. 2, November, 1898; New York.

"Transactions of the Ottawa Literary and Scientific Society," No. 1, 1897–8.

"Bulletin des Sciences Mathématiques," Tome XXII., November, 1898; Paris.

"Tōkyō Sūgaku-Butsurigaku Kwai Kiji," Maki No. VIII., Dai 3.

"Atti della reale Accademia dei Lincei—Rendiconti," Vol. VII., Fasc. 9, Sem. 2; Roma, 1898.

"Educational Times," December, 1898.

"Indian Engineering," Vol. XXIV., Nos. 17–20; Oct. 22–Nov. 12, 1898.

On the Null Spaces of a One-System and its Associated Complexes. *By* W. H. YOUNG. Received November 3rd, 1898, and subsequently, in revised form, February 25th, 1899. Read November 10th, 1898.

A one-system lying in an odd space S_{2m-1} is known to be reducible in general to m one-vectors. Of these m one-vectors, r can be chosen to lie in any odd space S_{2r-1} of perfectly general position, the $S_{2m-1-2r}$ containing the remaining $(m-r)$ one-vectors being then determined, as also are the systems in the S_{2r-1} and $S_{2m-1-2r}$ respectively. A slightly different theorem holds good for even spaces.* I here occupy myself with the theory of the spaces for which these theorems require modification. Such spaces, which I have called the null spaces of the one-system, are of several species. An S_r, where r is less than m, may have any species from 1 to $\frac{r}{2}$ or $\frac{r+1}{2}$ inclusive, according as r is even or odd. An S_r of maximum species I call a *thoroughly* null space, since in this case none of the m one-vectors can be chosen to lie in it. The null lines of the one-system generate a linear complex, and conversely, given a linear complex, we may construct it by means of a one-system. The thoroughly null spaces are the *vollständige Raume* of the associated linear complex discussed by S. Kantor,† and generate, as he has shown, a linear $\infty^{(n-\frac{3}{2}r)(r+1)}$ complex. More generally, the null S_r's of species p generate a linear $\infty^{(n-r)(r+1)-\frac{1}{2}p'(p'+1)}$ complex, where p' is $2p$ or $(2p-1)$ according as r is even or odd.

The methods employed are those indicated in Grassmann's *Ausdehnungslehre*, an English exposition of which has been given by E. Lasker in papers on "The Geometrical Calculus," published in the *Proceedings* of this Society, Vol. XXVIII. All that is required for the present paper will, however, be found in the paper by me "On Flat-Space Coordinates" (*infra*, pp. 54–69).

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* *Cf.* "On Systems of One-Vectors in Space of n Dimensions," *Proc. Lond. Math. Soc.*, Vol. XXIX., Theorems VII. *b* and *c*.

† "Allgemeine Theorie der linearen Complexe," 1897, *Crelle's Journal.*

2. *Notation.*

Let $a_{12}, a_{13}, \ldots, a_{1,n+1}, a_{23}, a_{24}, \ldots, a_{2,n+1}, \ldots, a_{n,n+1}$ be a system of quantities, and let

$$a_{ij} = -a_{ji}.$$

We adopt the following notation

$$12 \equiv [12] \equiv a_{12},$$

$$[1234] \equiv 12.34-13.24+14.23,$$

$$\vdots$$

$$[123 \ldots 2r] = 12.34 \ldots [(2r-1)\,2r]-13.24 \ldots [(2r-1)\,2r]+\&c.,$$

where the rule for writing down the right-hand side is as follows:—Every possible interchange of numerals is made in the first term, every such interchange being accompanied by a change of sign; the right-hand side then consists of the sum of all terms so obtained.

It follows at once from the definition that

$$[12 \ldots 2r] = [12][34 \ldots 2r]-[13][24 \ldots 2r]+\ldots$$
$$\ldots+[1, 2r][23 \ldots 2r-1],$$

with a number of similar expansions.

It will be convenient to speak of the *dimensions of a square bracket;* thus r will be said to be the dimensions of the square bracket on the left-hand side of the above equation. It is evident that, if all the square brackets of any the same dimensions vanish, all those of higher dimensions also vanish.

When n is odd and equal to $2m-1$ the square of the square bracket o dimensions m is the skew determinant formed in the usual way from the quantities a_{ij}. When n is even the determinant is, of course, identically equal to zero. The square of any other square bracket, whether n be odd or even, is a minor of this determinant. From the known properties of determinants, we deduce a variety of theorems, *e.g.*, if, n being odd, we denote by b's the square brackets of dimensions $(m-1)$ in the a's, the square brackets of dimensions $(m-1)$ in the b's are proportional to the square brackets of dimensions *one* in the a's, *i.e.*, to the a's themselves. We have, in fact, the identical equation

$$[34 \ldots 2m]_b \equiv \{[12 \ldots 2m]_a\}^{m-2}\, a_{12},$$

where we have written subscripts to call attention to the letters used in forming the square brackets. This theorem will be of use in the sequel.

3. *Covariants of a System of One-Vectors.*

Consider any system of one-vectors, not necessarily reduced to a canonical form. Take every possible set of r non-intersecting one-vectors: each such set determines a $(2r-1)$-vector, defined, without ambiguity, by the $2r$-pyramid having those r one-vectors for non-intersecting edges. Proceeding thus, we obtain a $(2r-1)$-system, easily seen to be covariant with respect to the original one-system. Suppose the original one-system changed in any manner whatever to an equivalent one-system; then the laws of the composition and resolution of vectors show that at every step of the work the $(2r-1)$-system changes into an equivalent $(2r-1)$-system. Here r must be such that $(2r-1)$ is less than n. If $(2r-1)$ be equal to n, a similar process will give us a number of scalar quantities, whose sum remains invariable, which is therefore an invariant of the system.

Again, take any fixed q-vector, external to the system, and let q and r be such that $(2r+q)$ is less than n, and form $(2r+q)$-pyramids by joining up the q-vector to all the $(2r-1)$-vectors of the $(2r-1)$-system. We thus get a $(2r+q-1)$-system, which possesses also covariant character with respect to the original system. If $(2r+q)$ be equal to n, we have a system of scalar quantities whose sum is constant for the same q-vector and the same one-system.

4. *Coordinates of a Covariant System.*

There are two modes in which we find it convenient to change a one-system. In the first, we replace each one-vector by components along the edges of a fundamental $(n+1)$-pyramid. We thus get the system replaced by single one-vectors along the edges; the ratios of these to the one-vectors denoted by the edges themselves, taken in the order 12, 13, ..., 23, 24, ... always in ascending order of numerals, we shall call the coordinates of the one-system, and denote by the symbols a_{ij} of § 2. In the same way the coordinates of a $(2r-1)$-system are thus defined; replace the $(2r-1)$-system by $(2r-1)$-vectors in the S_{2r-1} faces of the fundamental pyramid; the coordinates of the system are defined to be the ratios of these vectors to the $(2r-1)$-vectors represented by the fundamental $2r$-pyramids in those faces, the vertices being taken in definite order, *e.g.*, $(12 \dots 2r)$, so that the order of the numerals is always ascending.

Suppose, then, the one-system replaced in the first mode. Then we know the covariant $(2r-1)$-system is equivalent to that got by

combining the one-vectors along every r non-intersecting edges of the fundamental pyramid. This gives us the coordinates of the $(2r-1)$-system.* For example, in the S_{2r-1} whose vertices are numbered 1, 2, ..., $2r$, the magnitude of the $(2r-1)$-vector bears to that of the corresponding fundamental $(2r-1)$-vector a ratio which is expressed by means of the square bracket $[12 \ldots 2r]$, and similarly for the other coordinates. That is, *the coordinates of the* $(2r-1)$-*system are the square brackets of dimensions* r. Since these coordinates depend only on $\frac{1}{2}n(n+1)$ quantities, while those of the general $(2r-1)$-system are $\dfrac{\overline{n+1}!}{2r!\,\overline{n+1-2r}!}$ in number, it is evident that these systems are of very special types, except when $r = m-1$, that is, except for the $(n-2)$-system.

5. *Degeneration of One-Systems.*

Now let us use the second mode of reduction, in which we replace the one-system by the minimum number, say k, of equivalent one-vectors; this we may call the reduction to a canonical form. Taking the k one-vectors r at a time, we get a system of $\dfrac{k!}{r!\,\overline{k-r}!}$ $(2r-1)$-vectors, forming a system of which the expressions already obtained are the coordinates.

It is thus evident that, when r is greater than k, the square brackets of dimensions r are all zero.

We can at once deduce the conditions that a given one-system should degenerate one or more times. We can, for example, write down the conditions that the one-system should be equivalent to k one-vectors, where k is any integer less than m.† In this case, and in this case only, the covariant system of $(2k+1)$-vectors does not exist. *Thus the necessary and sufficient conditions that a system should be equivalent to k one-vectors are that the square brackets of dimensions* $(k+1)$ *should vanish, and those of dimensions k should not vanish.* Further the $(2k-1)$-system is equivalent to a single vector, namely, that determined by the k one-vectors, and *the coordinates of the* S_{2k-1} *in which the k one-vectors lie are given by the square brackets of dimensions k.*

6. *Classification of Linear S_{n-2} Complexes.*

The equation to a linear S_{n-2} complex involves the $\frac{1}{2}n(n+1)$ coordinates of an S_{n-2}. These are connected by the well-known

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* *Cf. infra*, § 16, p. 67. † *Cf.* Lasker, *loc. cit.*

quadratic relations whose number is the number of combinations of $(n+1)$ things taken four at a time. The equation also contains $\frac{1}{2}n(n+1)$ arbitrary coefficients; these may be regarded as the homogeneous coordinates a_{ij} of a certain one-system defined by them. The equation expresses the fact that the S_{n-2}, whose coordinates it contains, is such that the moment* of the one-system about it vanishes. The nature of the complex depends then on the nature of the system of one-vectors. We are thus led to a classification of the S_{n-2} complexes, which may be said to degenerate† 1, 2, ..., k times, according as the auxiliary one-system is reducible to $(m-1)$, to $(m-2)$, ..., to $(m-k)$ one-vectors. These different classes of S_{n-2} complexes will accordingly be characterized by the vanishing of the square brackets of corresponding order.

7. *Degenerate S_{n-2} Complexes.*

The consideration of complexes in even space, and of degenerate complexes in general, is easily seen to be reducible to that of undegenerate complexes in odd space. Thus suppose the one-system equivalent to k one-vectors lying in an S_{2k-1}. The complex consists of all those S_{n-2}'s whose moment about the one-system is zero, and thus evidently includes all those S_{n-2}'s which contain the S_{2k-1}, or which meet it in an S_{2k-2}. Consider, however, an S_{n-2} which meets the S_{2k-1} in an S_{2k-3}; the necessary and sufficient condition that its moment about the one-system should vanish is that in the S_{2k-1} the moment of this S_{2k-3} should vanish, that is, that the S_{2k-3} of intersection should belong to the undegenerate complex in the S_{2k-1} determined by the k one-vectors. Thus we may generate the original complex (undegenerate if n be even and equal to $2k$), as follows:—*Describe the undegenerate S_{2k-3} complex of the S_{2k-1} which corresponds to the k one-vectors. Draw through its S_{2k-3}'s all possible S_{n-2}'s; further, if n be not equal to $2k$, complete the system by drawing all possible S_{n-2}'s through the S_{2k-1}, and through the S_{2k-2}'s contained in the S_{2k-1}; if n be equal to $2k$, the system has to be completed by taking all the S_{n-2}'s in the S_{2k-1}.*

8. *Classification of Linear Line Complexes.*

The results of the preceding two articles may be at once applied, by means of the principle of duality, to the theory of linear line complexes. In fact, we may map off such a complex unit by unit on to an S_{n-2} complex, corresponding line and S_{n-2} having the same

* See *infra*, §§ 5, 6, pp. 57–59. † See note, p. 38.

coordinates, and the line complex and the dual S_{n-2} complex the same equation. Regarded as the equation to a line complex, the equation expresses the fact that the moment about every line of the complex of a certain $(n-2)$-system vanishes. We have only to modify the phraseology in the usual way. Thus, corresponding to an S_{n-2} complex degenerate k times, we have a line complex degenerate the like number of times.

In odd space $(n = 2m-1)$ corresponding to the auxiliary one-system of the S_{n-2} complex, lying in an $S_{2m-2k-1}$, we have an auxiliary $(n-2)$-system of the line complex, passing through an $S_{2\ -1}$. It is usual to call such an S_{2k-1} the *centre* of the complex; its coordinates are the same as those of the $S_{2m-2k-1}$ of the auxiliary one-system of the S_{n-2} complex, and may therefore be at once written down.

In even space $(n = 2m)$, we have an auxiliary $(n-2)$-system of the line complex, passing through an S_{2k} centre, where k may be zero, in which case we have the undegenerate complex with its point centre.

Whether the space be odd or even we have the following result:—

The conditions that a linear line complex should be degenerate k times are that all the square brackets of dimensions $(m-k+1)$ formed from the coefficients in the equation should vanish, and those of dimensions $(m-k)$ should not vanish, and the coordinates of the centre of such a degenerate complex are given by the square brackets of dimensions $(m-k)$. This is true for the undegenerate complex in even space $(n = 2m)$ if we put k equal to 0.*

9. *Degenerate Line Complexes and the Undegenerate Line Complex in Even Space.*

Consider a linear line complex degenerate k times, where, when $k = 0$, we understand the undegenerate line complex in even space. The complex has an $S_{n-2m+2k}$ centre. The auxiliary system of $(n-2)$-vectors then reduces to $(m-k)$ $(n-2)$-vectors, passing through the $S_{n-2m+2k}$ centre. Making use of the method of sections,† and cutting by an $S_{2m-2k-1}$, which does not meet the S_{2k-1}, we obtain a system of $(2m-2k-3)$-vectors lying in the $S_{2m-2k-1}$, whose moments about a unit vector in the S_{2k-1} are the original $(n-2)$-system. Form the

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* *Cf.* Segre, "Ricerche sulle omografie e sulle correlazione," *Memorie R. Acc. Torino*, 1885. It should be noted that in Segre's terminology "specialized" does not correspond exactly to "degenerate"; a line complex is specialized q times when it has an S_{q-1} centre. A line complex in even space is *always* specialized an odd number of times. The point of view from which we regard line complexes leads naturally to the adoption of language differing from that already in use.

† See *infra*, § 4, p. 57.

linear complex composed of the lines lying in the $S_{2m-2k-1}$, whose moment about this system vanishes. Every such line is evidently a line of the original complex. Conversely, by taking all possible $S_{2m-2k-1}$'s, we shall obtain all the lines of the original complex. It is, however, unnecessary to take more than one $S_{2m-2k-1}$. In fact we have merely to join up to the $S_{n-2m+2k}$ centre all the lines of the subsidiary line complex in the $S_{2m-2k-1}$. We thus get a complex of $S_{n-2m+2k+2}$'s, such that every line in every $S_{n-2m+2k+2}$ of it is a line of the original complex. The proof of this is evident, if we reflect that every line that meets the centre is a line of the complex, and therefore in every such $S_{n-2m+2k+2}$, we can construct a fundamental $(n-2m+2k+3)$-pyramid, every S_1 edge of which is a line of the complex, that is, is such that the moment about it of the auxiliary $(n-2)$-system vanishes. Any one-vector in the $S_{n-2m+2k+2}$ may be replaced by components along the edges of this pyramid; therefore the moment of the $(n-2)$-system about it is also zero. We may add that every line of the complex is obtained in this way, and obtained only once.

10. *The Covariant One-System of an $(n-2)$-System.*

There is a more picturesque and often more convenient method of passing from a linear line complex to a one-system than the method of duality.

We have seen that the covariant r-system of an undegenerate one-system in odd space is, in every case but one, special in character. The exception is when r is equal to $(n-2)$.

Every undegenerate $(n-2)$-system in odd space is, in fact, the covariant $(n-2)$-system of a certain one-system. We proceed to show how to construct the one-system, and to find its coordinates. Such an $(n-2)$-system is reducible to m $(n-2)$-vectors. We may conceive it accordingly represented by such a canonical set. Every $(m-1)$ of these vectors intersect in a straight line. Thus we obtain m straight lines, such that every $(m-1)$ of them determine the S_{n-2} of one of the canonical set.

Now suppose one-vectors whose magnitudes are $x_1, x_2, \ldots, x_m$ taken along these lines. If m equations hold of the form

$$x_2 x_3 \ldots x_m = A_1,$$

where A_1 is the ratio of the first $(n-2)$-vector to the $(n-2)$-vector obtained by taking unity for each x, then we shall have the required one-system. Evidently these equations are satisfied by

$$x_1 : x_2 : \ldots : x_m = \frac{1}{A_1} : \frac{1}{A_2} \ldots : \frac{1}{A_m}.$$

It is convenient to know the coordinates of the one-system when those of the $(n-2)$-system are given. Let them be denoted by a's, and those of the $(n-2)$-system by b's. Then, by § 4, we have to solve equations of the form

$$\lambda\,[12]_b \equiv \lambda b_{12} = [34 \ldots 2m]_a.$$

By § 2, the solutions of these equations are of the form

$$[12]_a \left\{[12 \ldots 2m]_a\right\}^{m-2} = [34 \ldots 2m]_b \lambda^{m-1}.$$

In other words, the homogeneous coordinates of the one-system may be taken to be the square brackets of order $(m-1)$ *in the* b*'s.*

This one-system may be called the covariant one-system of the $(n-2)$-system. Its covariant systems will also be covariant with respect to the $(n-2)$-system; and, without dwelling on the proof, we may assert that a covariant $(2r-1)$-system has for coordinates the square brackets of dimensions r in the a's or $(m-r)$ in the b's.

If the system be a degenerate one in odd space or degenerate or undegenerate in even space, the above requires modification; the one-system obtained is then a concomitant of the second type described in § 3, having an element of arbitrariness. Let us consider an $(n-2)$-system, degenerate k times, where, for $k=0$, we have the undegenerate complex in even space. The $(n-2)$-system is then equivalent to $(m-k)$ $(n-2)$-vectors all intersecting in an $S_{n-2m+2k}$. By the method of sections obtain a set of $(2m-2k-3)$-vectors in an $S_{2m-2k-1}$ not intersecting the $S_{n-2m+2k}$. This latter set is undegenerate and in odd space, and may therefore be treated by the method explained above. We thus get an undegenerate one-system in the $S_{2m-2k-1}$, which has this property: taking all but one of the one-vectors, and building up with these and with the complementary $(n-2m+2k)$-vector an $(n-2)$-vector, the system so obtained is the original set. The element of arbitrariness in this one-system is the choice of the $S_{2m-2k-1}$, which is only restrained not to cut the $S_{n-2m+2k}$, and is therefore any $S_{2m-2k-1}$ of perfectly general position.

11. *Construction of a Linear Line Complex.*

Anticipating the definition and discussion of §§ 12, 13, we may here insert simple constructions as resulting from the preceding article. *To construct an undegenerate linear line complex from its equation. Form the auxiliary* $(n-2)$*-system, and deduce the covariant one-system. The null-lines of the one-system are the lines of the complex.*

Next, to construct a linear line complex having an $S_{n-2m+2k}$ centre. As in § 9, obtain a system of subsidiary $(2m-2k-3)$-vectors lying in an $S_{2m-2k-1}$, not cutting the centre. Form the covariant one-system of this subsidiary system. Take all the null lines of this one-system, and join up these null lines to the centre, thus forming a complex of $S_{n-2m+2k+2}$'s. Every line in every $S_{n-2m+2k+2}$ of this complex is a line of the original complex.

Conversely, given a one-system, degenerate k times, and therefore equivalent to $(m-k)$ one-vectors, lying in an $S_{2m-2k-1}$, we can construct a linear complex, degenerate the same number of times, as follows :—Take an $S_{n-2m+2k}$, not intersecting the $S_{2m-2k-1}$, and join up to the null lines of the one-system in the $S_{2n-2k-1}$; all the lines in the $S_{n-2m+2k+2}$'s so obtained constitute the complex required.

12. *Null Spaces.*

We shall now confine our attention to odd space $(n = 2m-1)$, and to undegenerate systems, unless the contrary is stated or obviously implied. The necessary modifications for even space, or when the system is degenerate, may easily be made.

In the paper on one-vectors it was shown that, in the canonical form, one of the m representative one-vectors may be made to act along any straight line of perfectly general position, and that, more generally, r of them can be made to lie in any S_{2r-1} of perfectly general position. It is, however, evident that for special positions of the S_{2r-1} this would not be possible.* In fact, when it is possible to choose r of the m one-vectors of the canonical form in our S_{2r-1}, the remaining $(m-r)$ one-vectors will define a $(2m-2r-1)$-vector, whose moment about the S_{2r-1} does not vanish, while all the other $(2m-2r-1)$-vectors, got by taking together $(m-r)$ one-vectors of this canonical form, have a zero moment about the S_{2r-1}. For the theorem to be true the S_{2r-1} must then not be such that the moment of the covariant $(2m-2r-1)$-system about it is zero. If this moment be zero, we shall call the space in question a *null space*.

It is evident from § 10 that the null spaces of a one-system may equally well be called the null spaces of the $(n-2)$-system with which it is associated, and *vice versa*.

* Thus in Theorem v. (*b*) the construction fails if the chosen line through O be one of the ∞^3 which meet the S_3 determined by the second and third one-vectors. Since O is arbitrary, this shows that the construction fails for ∞^7 of the ∞^8 straight lines in S_5, which are cases of exception and, in accordance with the definition to be presently given, will be called *null lines*.

13. *Null Lines.*

Confining our attention first to null lines, we see that a null line may be defined in two equivalent ways:

(1) *As a line along which a one-vector of the canonical form cannot be made to act.*

(2) *As a line such that the moment of the covariant $(n-2)$-system about it vanishes.*

The equivalence of these two definitions follows from the mode in which we show that along a straight line of perfectly general position one of the one-vectors of the canonical form may be made to lie. The proof only fails when the arbitrary line lies in the fixed S_{n-1}, through the arbitrarily chosen point, and the S_{n-2} of the remaining one-vectors. Adopting a convenient term, we may say: the construction fails always, and fails only, when the line lies in the "polar" S_{n-1} of one of its points.

From this definition of polar it is evident that the polar of every point on such an exceptional line contains the line. Choosing some definite line through the arbitrary point along which one of the one-vectors should act, we have a fixed S_{n-2}, in which the remaining $(m-1)$ one-vectors lie. This S_{n-}, lying in the polar S_{n-1}, cuts any null line through the arbitrary point, and through this second point one of the remaining one-vectors can be made to pass. Thus every null line, according to the first definition, is such that two of the one-vectors may be made to meet it; it at once follows that it is null according to the second definition. That the second definition involves the first has virtually been shown in the preceding article. For, if a one-vector could act along a line, the moment of the covariant $(n-2)$-system about it could not vanish.

It is easily seen that *the moment about any straight line of the covariant $(n-4)$-system, and of systems of lower order, can none of them vanish.* For, by § 4, if any one of these vanish, all the covariant systems of higher order also have zero moment round the line. Hence the covariant $(n-2)$-system has zero moment round the line, so that the line is a null line. Choosing two of the one-vectors to meet it, a third cannot do so, for three such would lie in S_4, and be reducible to two. Now the moment of the $(n-4)$-system about a line meeting two of the one-vectors is evidently not zero. Thus the theorem is proved.

14. *Thoroughly Null Spaces.*

We shall use the expression *thoroughly null space* to denote a space which has no lines in it other than null lines. We easily see that such spaces exist by the following construction :—

THEOREM 1.—*If $(r+1)$ points be taken, one on each of as many distinct one-vectors of any canonical set of m one-vectors, these always determine a thoroughly null S_r.*

For they are the vertices of an $(r+1)$-pyramid, of which every edge is a null line. Consider then any other line in the S_r. A unit one-vector along it may be replaced by components along the edges of the $(r+1)$-pyramid, and its moment about the covariant $(n-2)$-system is the sum of the moments of these components. But the moment of each component is zero, for its line of action is a null line; therefore the moment of the line in question is zero, and the line is also a null line, as was to be proved.

THEOREM 2.—*If S_r be a thoroughly null space, the polar of every point in it contains the whole S_r.*

For the polar contains all the null lines through its pole.

THEOREM 3.—*If S_r be a thoroughly null space, we may always arrange that $(r+1)$ of the m one-vectors should meet it; more cannot meet it.*

For, through any point A of the S_r we may make one of the one-vectors pass. The remaining $(m-1)$ one-vectors then lie in the S_{n-1} polar of A; they also lie in the polar of any other point B on the line of action of the first one-vector. The polar of A contains the null line S_r, and the polar of B does not. Thus the S_{n-2}, in which the $(m-1)$ remaining one-vectors lie, does not contain the S_r, but meets it in an S_{r-1}, which is, of course, thoroughly null. Repeating this process r times, we get r one-vectors passing through as many points of the S_r, and the remaining $(m-r)$ lying in a space which intersects the S_r in a point. Through this point one of the remaining one-vectors may be made to act. This demonstrates the theorem.*

It is to be remarked that here $r+1 \leqslant m$.

It will appear subsequently that there are no thoroughly null spaces of dimensions higher than $(m-1)$.

THEOREM 4.—*A thoroughly null S_r is such that the moment about it of the covariant $(n-2r)$-system vanishes.*

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* It is otherwise obvious that an S_r cannot meet more than $(r+1)$ of the one-vectors. If it met $(r+2)$, these would reduce to $(r+1)$.

This follows from Theorem 3. For a thoroughly null S_r meets $(r+1)$ of the m one-vectors, and therefore meets the spaces determined by every $(m-r)$ of them.

THEOREM 5.—*Conversely an S_r such that the moment about it of the covariant $(n-2r)$-system vanishes is thoroughly null.*

The proof of this theorem is contained in that of Theorem 7, § 15, of which it is a special case.

15. *On the Species of Null Spaces.*

An S_{2r-1} or an S_{2r} of perfectly general position can be made, as we know, to contain r of the one-vectors, and, evidently, no more. We have seen, on the other hand, that spaces exist which cannot be made to contain even one of the one-vectors. We are thus led to a classification of spaces, with respect to a given one-system, according to the number of one-vectors of the canonical form which they can be made to contain.

We shall say that an S_{2r-1}, or an S_{2r}, is a null space of species p if the maximum number of one-vectors which it can be made to contain is $(r-p)$.

If the dimensions of the space in question be less than $(m-1)$, the last article shows that p can attain the maximum r, the space being then what we call thoroughly null. The maximum value of p when the space is of dimensions higher than $(m-1)$ is given by the following theorem:—

THEOREM 1.—*In every S_{n-r} where r is less than m, at least $(m-r)$ of the one-vectors can be made to lie.*

Since the one-system can be replaced by a one-vector through any chosen point, and $(m-1)$ in any S_{n-1} not containing the point, the theorem is obviously true when $r=1$. Suppose it proved for all values of r up to $(k-1)$: we proceed to prove it by induction, when r is equal to k.

Take any S_{n-k}, and through it pass an S_{n-k+1}; then $(m-k+1)$ of the one-vectors can be chosen in this S_{n-k+1}. These determine an $S_{2m-2k+1}$ in the S_{n-k+1}, which is cut by the S_{n-k} (unless the latter contains it) in an S_{2m-2k}. In this S_{2m-2k} of $S_{2m-2k+1}$ we may, by hypothesis, put $(m-k)$ of the $(m-k+1)$ one-vectors. Thus $(m-k)$ of the one-vectors can be put in the S_{n-k}, as was to be proved.

Thus the maximum value of p for an S_{n-2r} is the same as for an S_{2r-1}; viz., it is r. Similarly, it is r for an S_{n-2r-1}; that is, the same as for an S_{2r}.

THEOREM 2.—*If, in an S_{n-r} where r is less than m, $(m-r)$ of the one-vectors, and no more, can be made to lie, all the remaining r one-vectors can be made to meet it.*

For the remaining r one-vectors lie in an S_{2r-1}, which is cut by the S_{n-r} in an S_{r-1}. Since none of the one-vectors can be chosen in this S_{r-1}, it is thoroughly null. Hence, by Theorem 3 of the last article, all these r one-vectors can be made to meet it.

THEOREM 3.—*If, in an S_{n-r} where r is less than m, $(m-r+k)$ of the one-vectors, and no more, can be made to lie, then $(r-2k)$, and no more, can be made to meet it.*

The remaining $(r-k)$ one-vectors determine an $S_{2r-2k-1}$, which is met by the S_{n-r} in an S_{r-2k-1}. As, by hypothesis, none of the $(r-k)$ one-vectors can be made to lie in this, the S_{r-2k-1} is a thoroughly null-space for the system of $(r-k)$ one-vectors in the $S_{2r-2k-1}$. Therefore $(r-2k)$ of these one-vectors may be made to meet it. The remaining k one-vectors will not meet it.

THEOREM 4.—*If, in an S_r where r is less than m, k of the one-vectors, and no more, can be made to lie, then, of the remaining $(m-k)$ one-vectors, $(r-2k+1)$ may be made to meet the S_r, and no more.*

For the $(m-k)$ one-vectors lie in an S_{n-2k}, intersecting the S_r in an S_{r-2k}, which is, by the hypothesis, thoroughly null for the system in the S_{n-2k}. Hence $(r-2k+1)$ of these $(m-k)$ one-vectors, and no more, can be made to meet the S_{r-2k}, which proves the theorem.

From the last three theorems it follows that every null space may be constructed by means of a suitable choice of the canonical set of m one-vectors, as follows :—

THEOREM 5.—*To construct an S_r of species p, take a suitable canonical set of m one-vectors, choose 2p points if r be odd, and $(2p+1)$ if r be even, on as many different one-vectors, and complete the space by taking sufficient of the remaining one-vectors to give an S_r.*

We have still to show that every space constructed in this way is of species p ; that is, we have to show that a space so constructed has the maximum number of one-vectors in it, and could not, by another choice of the canonical form, be made to contain more. This follows from our next theorem.

THEOREM 6.—*If S_{2r-1} be of species p, the moment about it of the covariant $(n-2r-2p+2)$-system vanishes, as does that of any higher and no lower system.*

If S_{2r} be of species p, the moment about it of the covariant $(n-2r-2p)$-system vanishes, as does that of any higher and no lower system.

These follow from Theorem 5.

The covariant properties possessed by these spaces indicate that a space which when constructed with a particular set of one-vectors contains λ and meets μ in the manner explained could not contain more than λ, and therefore meet less than μ, by means of any other canonical form, for then we should have different covariant systems with zero moment about the space. In fact, if a different choice of the canonical form could give the numbers λ' and μ', we must have, first of all,

$$2\lambda+\mu = 2\lambda'+\mu' = \text{dimensions of the space.}$$

$(\lambda+\mu)$ will therefore necessarily alter, and with it the covariant system whose moment vanishes.

THEOREM 7.—*If an S_{2r-1} be such that the moment of the covariant $(n-2r-2p+2)$-system about it vanishes, and not that of any lower system, the S_{2r-1} is of species p.*

And, if an S_{2r} be such that the moment of the covariant $(n-2r-2p)$-system about it vanishes, and not that of any lower system, the S_{2r} is of species p.

Any other species is, in fact, inconsistent with the Theorem 6.

Summing up, we have proved the identity of two definitions of species, one respecting the number of one-vectors which can be made to lie in a space, and the other respecting the covariant systems which have zero moment about the space. We have also given a geometrical construction for a null space of any species.

16. *Polarity.*

We have hitherto only defined the polar of a point. Next to define the polar of a straight line.

It may be proved that the polar S_{n-1}'s of every point on a straight line intersect in an S_{n-2}; this we call the polar S_{n-2} of the straight line. If the line be not a null line the theorem on which the definition depends is obvious, the S_{n-2} being that determined by the remaining one-vectors when one has been chosen to act along the line. Next, take a null line, and draw two one-vectors to meet it. These determine an S_3 containing the line, and the remaining one-vectors determine an S_{n-4}, not intersecting the S_3. Without disturbing the S_{n-4}, we can move the first pair of one-vectors about in their S_3, so that one of them intersects the null line in any desired point

of it. Thus the polar of any point of the null line contains the S_{n-2} determined by this S_{n-4} and the null line itself. Hence all the polars of points on a null line intersect in an S_{n-2} containing the null line.

A similar proof may be used to prove the existence and properties of the polar S_{n-2r} of an S_{2r-1}. For let the S_{2r-1} be of species p. Draw $(r-p)$ one-vectors to lie in it and $2p$ to meet it. These determine an $S_{2r+2p-1}$ containing the whole S_{2r-1}. The $2p$ points lie, as is seen in § 15, in a thoroughly null S_{2p-1}. The polar of any point in this S_{2p-1} contains of course the S_{2p-1} and also the $(r-p)$ one-vectors first chosen; hence it contains the whole S_{2r-1}. Thus without disturbing the S_{2p-1} we can move the other one-vectors about so that one of them passes through any chosen point of the S_{2r-1}. The S_{2p-1} lies therefore in the polar of every point of the S_{2r-1}. Further, without disturbing the remaining $(m-r-p)$ one-vectors, we may move the $(r+p)$ about in their $S_{2r+2p-1}$, so that one of them passes through any chosen point of the S_{2r-1}. Thus the polar of every point of the S_{2r-1} contains the S_{n-2r} determined by the $S_{n-2r-2p}$ of the one-vectors not meeting the S_{2r-1} and the thoroughly null S_{2p-1}. This we may call the polar S_{n-2r} of the S_{2r-1}. *Mutatis mutandis*, the proof holds for the S_{n-2r-1} polar of an S_{2r}.

COR. 1.—*An S_{2r-1} and its polar S_{n-2r} are of the same species p, and intersect in a thoroughly null S_{2p-1}.*

Similarly, for an even space S_{2r} of species p, we may show the theorem holds, provided $p>0$.

COR. 2.—The thoroughly null S_{m-1}'s are their own polars.

17. *Coordinates of Polars of thoroughly Null and other Spaces.*

Given the coordinates of a point, those of the S_{n-1} polar may be at once written down. We have merely to take the sum of the moments in every S_{n-1} face of the components of the point about the components of the system of $(n-2)$-vectors, *e.g.*,

$$p_{12\dots n} = p_1[23\dots n]-p_2[13\dots n]+\&c.$$

Given the coordinates of a line, those of its polar may be found as follows:—Assume the line is such that one of the one-vectors can be made to act along it, and let λ be the magnitude, and $p_{12}, p_{13}, \dots$ the coordinates of the one-vector so chosen. Then

$$a_{12}-\lambda p_{12} : a_{13}-\lambda p_{13} : \&c.,$$

are the coordinates of a system of one-vectors which is degenerate once; therefore the square bracket of dimensions m of these quantities

vanishes. Expanding in powers of λ, all the coefficients disappear except the constant term and the term involving the first power of λ. Thus we have the following equation defining λ:—

$$[12 \dots 2m] = \lambda \{p_{12}[34 \dots 2m] - p_{13}[24 \dots 2m] + \&c.\}.$$

This fails, it will be noticed, if the coefficient of λ vanishes. λ is then infinite; the line must therefore be a null line of the system. We may, however, deduce the result in this case also.

With this value of λ,

$$a_{12} - \lambda p_{12} : a_{13} - \lambda p_{13} : \&c.,$$

are the coordinates of the degenerate system. The S_{n-2} in which it lies is the required polar. Following, therefore, the rule of § 5, we form the square brackets of dimensions $(m-1)$; these are the coordinates required. If we go through the work, we find there is a great simplication, the coefficients of all powers of λ, except the first and the absolute term, vanishing. Thus, for instance,

$$p_{34\dots 2m} = [345 \dots 2m] - \lambda \{p_{34}[56 \dots 2m] - p_{35}[46 \dots 2m] + \&c.\}.$$

When the line is a null line the absolute term may be neglected in comparison to that involving λ, and the coordinate of the polar S_{n-2} of a null line may be taken to be the coefficient of λ itself.

We can see this more easily still geometrically. For, as the polar of a null line contains the null line, the polar S_{n-2} is that of the moment about the null line of the covariant $(n-2)$-system; the coordinate can therefore be written down in the usual way. In a similar way, the polar spaces of all thoroughly null spaces can be written down at once. Thus, to find the polar of a thoroughly null S_2. This is the space determined by the thoroughly null S_2 and the remaining $(m-3)$ one-vectors of the canonical form, when three have been made to meet it. That is, it is the space of the moment about the S_2 of the covariant $(n-6)$-system. More generally the polar of a thoroughly null S_r is that determined by the S_r, and the $(m-r-1)$ one-vectors which do not meet it, and is therefore the space of the vector representing the moment about the S_r of the covariant $(n-2r-2)$-system.*

18. *The Equations to the Complexes of Null Spaces.*

First consider the null lines. Their characteristic property is that the moment about them of the covariant $(n-2)$-system vanishes.

* See end of § 18. The indications are sufficient to enable the reader to find the coordinates of the polar of any space whatever. The results, being less simple, are not given here.

This moment is a scalar quantity; there is, therefore, only one condition that a line should be a null line; viz., it is

$$p_{12}[34 \dots 2m] - p_{13}[24 \dots 2m] + \&c. = 0,$$

or, in the notation of § 2, $\Sigma b_{ij} p_{ij} = 0.$

In the general case of a null S_{2r-1} of species p the characteristic property that the moment of the covariant $(n-2r-2p+2)$-system about it vanishes leads to several equations. In fact the moment of this system about a unit $(2r-1)$-vector in an S_{2r-1} gives rise, in the general case, to a system of $(n-2p+2)$-vectors; this system must, in our case, be in equilibrium. We thus have as many equations as such a system has coordinates, viz., the number of coordinates of a $(2p-3)$-system; that is, the number of $(2p-2)$-pyramidal faces in an $(n+1)$-pyramid. The coefficients in these equations are the square brackets of dimensions $(m-r-p+1)$.

In the case of a null S_{2r} of species p, the number of equations is the number of $(2p-1)$ pyramidal faces, and the coefficients are the square brackets of dimensions $(m-r-p)$.

It will be noted that the number of equations depends only on the species of the space, and not on its dimensions. The equations are not, however, all independent; they are linear equations, having syzygies between them.

To determine the dimensions of the entity formed by the S_r's of a given species, we must either determine the syzygies, and the syzygies between the syzygies, and so on, or else adopt another method. This we shall do in two subsequent articles (§§ 23, 24).

It will be noticed that, denoting by

$$\Theta = 0$$

the equation to the complex of null lines, Θ is the coefficient of λ in the equation of the preceding article determining λ; and that the coordinates of the polar S_{n-2} of a null line are

$$p_{34 \dots 2m} : p_{24 \dots 2m} : \&c. = \frac{\partial \Theta}{\partial a_{12}} : \frac{\partial \Theta}{\partial a_{13}} : \&c.$$

By Euler's theorem we then have

$$a_{12} p_{34 \dots 2m} + a_{13} p_{24 \dots 2m} + \&c. = 0,$$

the equation to the complex of null S_{n-2}'s of species 1, which verifies our result.

In a similar way, if

$$\Theta_{34\ldots 2m} = 0, \quad \Theta_{24\ldots 2m} = 0, \quad \&c.,$$

be the equations to the complex of null S_3's, where

$$\Theta_{34\ldots 2m} = p_{3456}\,[78\ldots 2m] - p_{3567}\,[48\ldots 2m] + \&c.,$$

the coordinates of the polar S_{n-4} of a null S_3 are

$$p_{56\ldots 2m} : p_{46\ldots 2m} : \&c. = \frac{\partial\Theta_{3456\ldots 2m}}{\partial a_{34}} : \frac{\partial\Theta_{3456\ldots 2m}}{\partial a_{35}} : \&c.$$

$$= \frac{\partial\Theta_{1256\ldots 2m}}{\partial a_{12}} : \frac{\partial\Theta_{1346\ldots 2m}}{\partial a_{13}} : \&c.,$$

and a variety of equivalent ratios.

Euler's theorem gives us then the equations to the complex of null S_{n-4}'s of species 2.

It is easy to generalize from the above and write down symbolically the coordinates of the polar of any thoroughly null space.

19. *Associated Complexes of a given Linear Complex.*

Suppose now the linear complex to be given, and let the coefficients in its equation be the quantities b. Then the preceding shows that there are a series of associated complexes, which may be written down as follows, making use of the known relations connecting the a's and the b's.

Linear Complex. $\quad \Sigma\, b_{ij}\, p_{ij} = 0.$

Complex of S_2's. $\quad \Theta_{2345\ldots 2m} = 0, \quad \Theta_{1345\ldots 2m} = 0, \ \&c.,$

where $$\Theta_{2345\ldots 2m} \equiv p_{234}\,[1234]_b - p_{235}\,[1235]_b + \&c.$$

Complex of S_3's. $\quad \Theta_{3456\ldots 2m} = 0, \quad \Theta_{2456\ldots 2m} = 0, \ \&c.,$

where $$\Theta_{3456\ldots 2m} \equiv p_{3456}\,[123456]_b - p_{3567}\,[123567]_b + \&c.$$

&c., &c.,

the coefficients corresponding to an S_r complex being always the square brackets of dimensions r.

These we shall call the *associated complexes of the given linear complex.* The original complex is defined by one equation; the associated complexes are each of them defined by several equations, with, more-

over, syzygies between them. The relation between the linear line complex and an associated S_r complex is this: *taking the lines of the original complex, we build up all possible S_r's; the S_r's so obtained constitute the associated complex in question.*

21. *On the Associated Complexes of a Degenerate Complex.*

Suppose the complex degenerate k times, so that it has an $S_{n-2m+2k}$ centre. The auxiliary $(n-2)$-system consists then of the system formed by joining the centre up to the $(2m-2k-3)$-vectors of a $(2m-2k-3)$-system lying in any $S_{2m-2k-1}$, which does not meet the $S_{n-2m+2k}$; this $(2m-2k-3)$-system being a covariant system of an undegenerate one-system in the $S_{2m-2k-1}$. We have to find the loci of the thoroughly null spaces of this $(n-2)$-system. We shall employ the method of duality. Corresponding to the degenerate $(n-2)$-system, we have a degenerate one-system lying in an $S_{2m-2k-1}$, having the same coordinates as those of the $(n-2)$-system. We can, at once, write down the coordinates of the covariant $(2r-1)$-system, where r may have any value from 1 to $(m-k)$ inclusive; they are the square brackets of dimensions r. Dually these are the coordinates of the covariant $(n-2r)$-system of the original $(n-2)$-system. That is to say, *the coefficients in the equations to the associated S_r complex, where r may have any value from* 1 *to* $(m-k)$ *inclusive, are the square brackets of dimensions r.*

22. *On the Arrangement of the Null Lines in the Null Spaces.*

The null lines in a null space, exception being of course made of the case in which the space is thoroughly null, form an entity whose dimensions are one less than the dimensions of the line space peculiar to the space. In fact, for a line to be a null line constitutes a single condition, viz., it must belong to a certain linear complex. The null lines of a space form therefore what we may call the "section" of the complex by this space. Considering for definiteness an S_{2r-1} of species p (where $2r \leqslant m$), let us consider the nature of the section for all values of p from 0 to $(r-1)$. When p is zero, that is, when the space is ordinary, r of the one-vectors can be made to lie in it, the system determined by these r one-vectors being a determinate one. The section in this case is composed therefore of the null lines of an undegenerate one-system of the S_{2r-1}; in other words, it is an undegenerate linear complex of that space.

Next suppose the S_{2r-1} to be of species $p>0$. In this case $(r-p)$

of the one-vectors may be made to lie in it, and $2p$ to meet it. These $2p$ determine by their intersection with the S_{2r-1} a determinate S_{2p-1}, which is thoroughly null. Every line in the S_{2r-1} which meets this S_{2p-1} is thoroughly null. For, take a point on such a line not in the S_{2p-1}; one of the $(r-p)$ one-vectors may be made to pass through it, and one of the $2p$ to pass through the point where it meets the S_{2p-1}, which shows that the line is a null line. Taking a definite set of $(r-p)$ one-vectors, these determine an $S_{2r-2p-1}$, and form an undegenerate one-system in it. The null lines of this system are null lines of the original one-system. Join up every such line to the S_{2p-1}; we thus get S_{2p+1}'s every line of which is a null line of the original one-system, and therefore a line of the linear line complex which it defines. It will be noted that, in virtue of § 9, the result we have arrived at is as follows :—

The section of the linear line complex of a one-system by a null space of species p is a linear line complex degenerate p times.

23. *Enumeration of the thoroughly Null Spaces.*

Let $f(n, r)$ denote the number* of thoroughly null S_r's for the general undegenerate one-system in S_n.

Then $\{f(n, r)-n+r\}$ is the number of such S_r's which pass through an arbitrary point. But all the thoroughly null S_r's which pass through a given point necessarily lie in the polar S_{n-1} of that point; and, taking any S_{n-2} of the polar S_{n-1} not passing through the pole, each such S_r will give a thoroughly null S_{r-1} of the S_{n-2}; and, *vice versa*, each thoroughly null S_{r-1} of that S_{n-2}, joined up to the pole, gives one of the null S_r's passing through the pole. Hence

$$f(n, r)-n+r = f(n-2, r-1).$$

Or, putting $n = 2m-1$, we may write

$$\phi(m, r) \qquad -\phi(m-1, r-1) = 2m-1-r;$$

therefore

$$\phi(m-1, r-1)-\phi(m-2, r-2) = 2(m-1)-1-r+1,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\phi(m-r+2, 2)-\phi(m-r+1, 1) = 2(m-r+2)-1-2;$$

further

$$\phi(m-r+1, 1) \qquad = 2(2m-2r)-1.$$

* We shall say that the number of S_r's is x when there are ∞^x of them.

These give at once

$$\phi(m, r) = (n-r)(r+1)-\tfrac{1}{2}r(r+1),$$

which is the number of the dimensions of the entity formed by the thoroughly null S_r's.*

As the total number of S_r's in S_n is $(r+1)(n-r)$, we see that the numerous equations obtained in § 18 are equivalent to $\frac{1}{2}r(r+1)$ only. Thus, for instance, the thoroughly null S_2's, the equations to which are $(n+1)$ in number, are only bound by three independent conditions.

24. *Enumeration of Null Spaces of any Species.*

We have seen that a null S_{2r-1} of species p intersects its polar in a thoroughly null S_{2p-1}; further, all the S_{2r-1}'s which pass through a thoroughly null S_{2p-1} and lie in the polar S_{n-2p} are of species p at least, and there are no others. Now in such an S_{2r-1} there is only one thoroughly null S_{2p-1} which has this unique position with regard to it. Hence the number of null S_{2r-1}'s of species p is equal to the number of S_{2r-1}'s which pass through a thoroughly null S_{2p-1}, and lie in its polar, *plus* the number of thoroughly null S_{2p-1}'s. Writing $2p-1 = p'$ and $2r-1 = r'$ for brevity, the required number is

$$(r'-p')(n-p'-1-r')+(n-p')(p'+1)-\tfrac{1}{2}p'(p'+1)$$
$$= (n-r')(r'+1)-\tfrac{1}{2}p'(p'+1),$$

which is $p(2p-1)$ less than the number of general S_{2r-1}'s. The equations to be satisfied by an S_{2r-1} of species p are therefore equivalent to $p(2p-1)$ independent conditions. The above reasoning holds for an S_{2r}, only that we have to write $p' = 2p$, $r' = 2r$ in accordance with § 15.

Combining this result with the rule of § 18, we see that *the null S_r's of species p generate a linear* $\infty^{(n-r)(r+1)-\frac{1}{2}p'(p'+1)}$ *complex*, where p' is $2p$ or $(2p-1)$ according as r is even or odd. These complexes may be said, in an extended sense, to be "associated" with the complex of null lines. The S_r's are not built up of complex lines, but the coefficients in the equations are still square brackets formed from the coefficients b_{ik}. Thus, for instance, the equation to the complex of null S_3's of species 1 is

$$p_{1234}[1234]_b-p_{1345}[1345]_b+\&c. = 0.$$

* *Cf.* S. Kantor, *loc. cit.*

*On Flat-Space Coordinates.** *By* W. H. Young.

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The theory of vector quantities in three dimensions is of comparatively simple character. This is in part due to the fact that they may all be treated as line-vectors. In space of higher dimensions this simplicity usually disappears. A couple, for example, can no longer be regarded as represented by a straight line. Its proper representation is now a plane, drawn through an arbitrary fixed point, and a certain magnitude regarded as attached to that plane. Again, the velocity of rotation of an n-dimensional body in space of n dimensions, is represented by a definite S_{n-2} and a magnitude attached to it.

I propose here to make use of the more general conception of vector which these considerations suggest. By this means I am able to extend to plane and flat-space coordinates the geometrical definition of line-coordinates given in a previous paper. The coordinates thus defined are shown to satisfy the well known quadratic equations, for which a geometrical form of statement is accordingly obtained. It is at the same time shown that these equations form a complete system. Throughout the work, which is geometrical in character, no particular prominence is given to coordinates of space of any one number of dimensions; but it appears finally that the usual determinantal expressions may be used for the coordinates here defined, provided we choose as point coordinates the coordinates of Möbius.

1. *Definitions of an r-Vector and an r-System.*

I consider a flat-space of n dimensions, and the flat-spaces, or hyper-spaces, contained by it. A flat-space of r dimensions I call *an*

* At the time of writing this paper I had not read Grassmann's *Ausdehnungslehre*. The student familiar with this treatise will recognize that the idea of an r-vector, and of the composition and resolution of r-vectors here expounded, is that which underlies Grassmann's work. (*Cf.* Mr. Lasker's paper in the *Proceedings* of the Society.) I have, however, been advised to publish the paper as it stands, as even those parts of it in which the connexion with the *Ausdehnungslehre* is most close may be useful to many as a help to the understanding of that difficult book.

r-space, or S_r. In an r-space I consider what I shall call an r-vector,* that is, a definite magnitude and sign connected with that r-space. An r-vector is conveniently denoted by an $(r+1)$-pyramid,† situated anywhere in the corresponding r-space; thus a 1-vector (force) by a bounded straight line, a 2-vector by a triangle in the corresponding plane, and so on. The magnitude is represented by the volume of the $(r+1)$-pyramid, measured in terms of any convenient unit; and the sense arises in each case from the order assigned to the angular points, there being, as is easily seen, in every case only two distinct orders.‡

We assume all the known properties of an $(r+1)$-pyramid, as, for instance, that any $(k+1)$ of its vertices determine an S_k, in which they are the vertices of a $(k+1)$-pyramid; and that, as is easily proved by induction, the volume of the $(r+1)$-pyramid is unaltered by replacing any $(k+1)$ of its vertices by $(k+1)$ other points, forming a $(k+1)$-pyramid of equal volume in the same S_k; while, if the volumes of the two $(k+1)$-pyramids be not equal, those of the two $(r+1)$-pyramids will be to one another in the ratio of the subsidiary $(k+1)$-pyramids. (*Pyramid theorem.*)

A system of r-vectors I shall call *an r-system.*

2. *Composition of r-Vectors and the Parallelogram Law.*

Consider two r-vectors, whose S_r's have an S_{r-1} in common, and therefore lie in an S_{r+1}. Take any r-pyramid in this S_{r-1}; then we may, by definition, represent the r-vectors by two $(r+1)$-pyramids having the r vertices of this r-pyramid in common. Let the remaining vertices be P and Q respectively. Let O be any point whatever in the common S_{r-1}. Find the resultant OR of OP, OQ according to the parallelogram law for 1-vectors.§ It may be proved that this point R, with the r-pyramid, determines an r-vector which depends

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* A couple does not represent the most general case of a 2-vector, in that the representative plane can be drawn through any arbitrary point we please. The r-vectors we here consider have reference to definite r-spaces.

† That is, the figure formed by $(r+1)$ points not lying in any space of lower dimensions. An $(n+1)$-pyramid chosen for reference will be called the fundamental $(n+1)$-pyramid.

‡ The interchange of two vertices changes the sense. Two $(r+1)$-pyramids in the same r-space are considered to have the same sense when, by mere distortion [without passing through the form of a degenerate $(r+1)$-pyramid of zero volume], and by motion through a *finite* portion of the r-space, the two $(r+1)$-pyramids can be brought into coincidence.

§ Bisect PQ in R' and produce OR' to R, so that $R'R = OR'$.

only on the two original r-vectors: this we define to be the *resultant* of the two r-vectors.

To prove the property, on which our definition depends, first keep P, Q fixed, and let O move anywhere on the common S_{r-1}. Then the point R' of bisection of PQ is also fixed, and R always lies in the S_r determined by R' and the common S_{r-1}, describing a fixed S_{r-1},* the reflexion in R' of the first S_{r-1}. Also, the volume of the pyramid determined by R is obviously twice that determined by R', and is therefore constant.

Next, keep O fixed, and the volume of the r-pyramid base, and alter the positions of one or both of P and Q. Then the minimum distances of P, Q from the common S_{r-1} remain constant; the same is therefore true of R' and of R. It follows that R describes the same S_{r-1} as before, since in both cases the minimum distance of R from the common S_{r-1} is constant and equal to twice that of R'.

Lastly, take a new r-pyramid base. Then the new vertices P_1, Q_1 may be taken on OP, OQ respectively, and such that

$$OP : OP_1 = OQ : OQ_1$$

is reciprocally proportional to the volumes of the corresponding r-pyramid bases. Then R'_1, the new position of R', will lie on OR, and $OR' : OR'_1$ will be in the same ratio. Hence the same is true of R and its new position R_1. It follows that, however, we modify the intermediate steps, consistent with the conditions of the problem, R remains in the same S_{r-1} lying in the S_{r+1}, and determines a pyramid of constant volume.†

Cor.—With the construction and notation of the preceding paragraph it is evident that the resultant of the two vectors represented by λ times the pyramid with P as vertex, and μ times that with Q as vertex, is completely represented by $(\lambda+\mu)$ times that with G as vertex, where G is the centre of mean position of λ at P, and μ at Q.

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* Parallel to the common S_{r-1}.

† The use of parallels is avoided in the text. We may, however, state the law as follows: through P draw an S_r parallel to that of the second r-vector (this contains, of course, the whole locus of P), and through Q an S_r parallel to that of the first vector. These two S_r's lying in an S_{r+1} intersect in an S_{r-1} (parallel to the common S_{r-1}). This last S_{r-1} is the locus of the vertex of the resultant pyramid. A corresponding statement of the parallelogram law of resolution can, of course, be given.

3. *Resolution of an r-Vector into Components.*

Take any S_{r-1} in the S_r of the vector; take any two S_r's through this S_{r-1}, lying with the given S_r in an S_{r+1}. Then we may replace the given r-vector by two components, one in each of these S_r's. Arrange that all but one of the vertices of the representative $(r+1)$-pyramid lie in the common S_{r-1}. Let R be the remaining vertex, and O any point of the common S_{r-1}. Through OR draw any plane in the S_{r+1}; this will cut the two chosen S_r's in straight lines. Decompose OR into components OP, OQ, along these straight lines. Then P and Q are the vertices of two pyramids representing the required components.

This decomposition is evidently unique.

4. *The Section Theorem.*

Given a system of r-vectors lying in an S_p, *and represented by* $\lambda_1, \lambda_2, \ldots$ *times appropriate* $(r+1)$*-pyramids, and a q-vector whose* S_q *does not intersect the* S_p; *the* $(q+r+1)$*-vectors represented by* $\lambda_1, \lambda_2, \ldots$ *times the* $(q+r+2)$*-pyramids whose vertices are obtained by taking in order the vertices of the* $(q+1)$*-pyramid with those of each* $(r+1)$*-pyramid in turn, will be subject to precisely the same laws of composition and resolution as the original r-vectors, and conversely.*

It is sufficient to prove this for a system of three vectors such that one is the resultant of the other two. But in this case the truth of the theorem is at once evident from the corollary to § 2.

The above suggests the convenience of introducing the term *point-vector*, to denote a point and a definite magnitude connected with it. The resultant of a system of point-vectors may then be defined to be a point-vector whose point is the centre of mean position of the magnitude situated at their corresponding points; the magnitude attached to it being the sum of the magnitudes of the system. With this definition the section theorem evidently holds when r is zero.

5. *Theory of Moments.*

If their spaces do not intersect, a p-vector and a q-vector may be said to have a moment. As in three dimensions, the vanishing of the moment must be the necessary and sufficient condition of intersection of the two spaces. We define the moment of the p-vector about the q-vector as the $(p+q+1)$-vector* represented by the

* If $n = p+q+1$, the vectorial character of the moment disappears, and we may regard the moment as a scalar quantity, whose magnitude is that of the pyramid.

$(p+q+2)$-pyramid got by taking first all the vertices of the representative $(p+1)$-pyramid, then all those of the $(q+1)$-pyramid, each in order.*

We see that this definition is justifiable, since the vector so obtained is, by the pyramid theorem, independent of the particular angular points chosen in the p-space and q-space respectively, provided only the $(p+1)$- and $(q+1)$-pyramids formed by them are of constant volume. The chief advantage of the introduction of this term is that the moment obeys a law, analogous to the ordinary law of moments in three dimensions, viz.,

If we decompose both the vectors into components, all lying however in the $(p+q+1)$-space determined by the two vectors, the algebraic sum of the moments of every component of the p-vector about every component of the q-vector is the same however the decomposition be performed, and equal to the moment of the p-vector about the q-vector.

To prove the law we only have to decompose one vector into two components: if the law be shown to hold, then, it must, by continual application, hold in the more general case.

Consider then two p-vectors whose S_p's have a common S_{p-1}. Let the representative pyramids have a common p-pyramid base, P and Q being the remaining vertices. Bisect PQ in R'. The resultant p-vector is represented by twice the pyramid on the same base with R' as vertex.

Take now a q-vector lying in the same $(p+q+1)$-space with all three p-vectors. The q-space may, or may not, intersect one or more of the p-spaces. If it intersects two of them, it must evidently intersect the third. Suppose first it intersects none of them. Complete the three $(p+1)$-pyramids by adjoining to them the vertices of the $(q+1)$-pyramid. That one of the three $(p+q+2)$-pyramids, so obtained, which has R' for vertex has for its volume half the sum of the volumes of the other two, since the minimum distance of R' from the common S_{p+q} base is half the sum of those of P and Q. But, by the pyramid theorem, the volume of the $(p+q+2)$-pyramid with R' as vertex is half that determined by substituting for R' the corresponding vertex of the actual pyramid representing the resultant

* I have found it convenient to drop the numerical multiplier $n!$ which is required to bring the definition into closer accordance with the usual theory of moments in three dimensions. It should also be noted that the moment of a $2p$-vector about a $2q$-vector has the opposite sign to that of the $2q$-vector about the $2p$-vector.

p-vector. This last $(p+q+2)$-pyramid represents the moment of the resultant p-vector about the q-vector, while the $(p+q+2)$-pyramids on the same base with P and Q respectively as remaining vertices represent the moments of the components. Hence the moment of the resultant is equal to the sum of the moments of the two components.

When the S_q intersects one of the S_p's the wording has to be slightly altered, since one of the $(p+q+2)$-pyramids has then zero magnitude, and one of minimum distances vanishes; the argument, however, still applies.

6. *Equilibrium of an r-System.*

I use the expression *moment of an r-vector about an* S_{n-r-1} to denote its moment about a unit vector in that S_{n-r-1}. By *the moment of an r-system about an* S_{n-r-1} I mean the sum of the moments of its r-vectors about that S_{n-r-1}.

If two r-systems be equivalent, so that, by repeated application of the parallelogram law, it is possible to pass from one system to the other, it is necessary that the moments of the two systems about every S_{n-r-1} should be equal.

A system which is equivalent to two equal and opposite r-vectors in the same S_r will be said to balance, or to be in equilibrium. If an r-system balance, its moment about every S_{n-r-1} must vanish.

THEOREM.—*A system of r-vectors lying in the S_r faces of the fundamental $(n+1)$-pyramid cannot balance.*

For, if so, the moment of the system about any S_{n-r-1} would vanish. Take any S_{n-r-1} face of the fundamental pyramid; this meets all the S_r faces except one. Hence the moment of the r-vector in that S_r face about the opposite S_{n-r-1} must vanish, which is impossible unless this r-vector itself vanish. Similarly every other r-vector must vanish.

COR.—Two systems of r-vectors lying in the S_r faces of the fundamental pyramid cannot be equivalent.

7. *Decomposition of an r-Vector into two Components, one lying in a given S_{n-1}, and one passing through a given point V outside that S_{n-1}.*

To effect this we only have to take the S_{r+1} determined by V and the S_r of our vector; this intersects the given S_{n-1} in an S_r, passing

through the S_{r-1} of intersection of the S_{n-1} with the S_r of the vector. Thus we have to decompose the r-vector into two components in the S_r so obtained, and in that through V and the S_{r-1} of intersection. We now proceed as in § 2.

We may notice that the S_r of the former component is the projection of the S_r of the vector from V on the S_{n-1}. It is sometimes convenient to adopt the following method of determining the magnitudes of the components. Choose the representative pyramid of the r-vector in such a way that r of its vertices lie in the S_{r-1} of intersection, and let R' be the projection of the remaining vertex R from V on the S_{n-1}. The first component is then represented by λ times the projected pyramid, and the second component by μ times the pyramid on the same r-pyramid base with V as remaining vertex, where

$$\lambda = \frac{VR}{VR'}, \quad \text{and} \quad \mu = \frac{RR'}{VR'},$$

so that

$$\lambda + \mu = 1.$$

This follows immediately from the corollary to § 2.

We may obtain a more general expression for the magnitude of the component in the S_{n-1} as follows :—

Taking any convenient representative $(r+1)$-pyramid, such that all but p of its vertices lie in the S_{n-1} (these determine of course a p-pyramid), the magnitude of the required component has the same ratio to the magnitude of the projected $(r+1)$-pyramid as that of the $(p+1)$-pyramid with V as vertex and base the p-pyramid to the $(p+1)$-pyramid with V as vertex, having the projected p-pyramid as base.

It is evident that this is the same as the ratio of the $(r+2)$-pyramid with V as vertex, on the chosen representative $(r+1)$-pyramid as base to the $(r+2)$-pyramid with V as vertex on the projected $(r+1)$-pyramid as base.

Now, if we choose a different representative pyramid, the first of these ratios varies inversely as the projected $(r+1)$-pyramid; also, by the pyramid theorem, since only the second $(r+2)$-pyramid alters in magnitude, the last of these ratios varies inversely as the projected $(r+1)$-pyramid. If, however, we choose a representative pyramid such that $p = 1$, the last ratio is that of $\dfrac{VR}{VR'}$, which proves the theorem.

8. THEOREM.—*An r-vector can be decomposed in one and only one way into components lying in the S_r's of the fundamental $(n+1)$-pyramid.*

Suppose such a reduction effected for S_r's and S_{r-1}'s in S_{n-1}; we proceed to show it can then be effected for S_r's in S_n.

By the last article we can replace a given r-vector by two, one through any vertex V, and the other in the opposite S_{n-1} face of the fundamental pyramid. By hypothesis the latter component may be replaced by components in the S_r's of that face.

Take as the representative pyramid of the former component one which has r of its vertices in the S_{n-1}, and V for the remaining vertex. The $(r-1)$-vector, represented by the r-pyramid base, lying, as it does, in the S_{n-1}, can, by hypothesis, be replaced by components in the S_{r-1}'s of the n-pyramid in the S_{n-1}. Therefore, by the section theorem, the r-vector may be replaced by components, represented by the $(r+1)$-pyramids, whose bases are the r-pyramids in the S_{r-1} faces of the S_{n-1}, and which have V for common vertex—that is, by components lying in S_r faces of the fundamental pyramid. Hence the reduction has been effected. Now we know that the theorem is true for point-vectors and line-vectors in S_2; therefore, by induction, the reduction is possible always. The uniqueness of the reduction follows at once from the corollary to § 6.

Such components I shall call *the component r-vectors in the S_r faces.*

9. *To show that the component r-vectors in the S_r faces in an S_p face, if $p>r$, or through an S_p face, if $p<r$, are equivalent to a single r-vector; and to find its position.*

In practice we may effect the reduction of the preceding article by repeated use of the construction of § 7, and the section theorem. It is then evident that the component r-vectors in the S_r faces of an S_p face are equivalent to a single r-vector, in the S_r obtained by projection on to the S_p face from the opposite S_{n-p-1}. This I shall call *the component r-vector in the S_p face.* Here p is greater than r.

It is similarly evident that, if p be less than r, the component r-vectors whose S_r's pass through an S_p face are equivalent to a single r-vector, lying in the S_r determined by that S_p, and the S_{r-p-1} of intersection of the opposite S_{n-p-1} with the S_r of the vector. This I shall call *the component r-vector through* that S_p face.

10. *Coordinates of an r-Vector and of an r-System.*

The ratio of the components of an r-vector in the S_r faces of the fundamental $(n+1)$-pyramid to the r-vectors represented by the corresponding $(r+1)$-pyramids of the faces, I call *the coordinates of the r-vector.** These coordinates are, of course, not all independent. It should be noted that the non-vanishing coordinates of the component r-vector in an S_p face are identical with the corresponding ones of the original vector.

An r-system can also be replaced in one and only one way, by r-vectors belonging to the S_r faces of the fundamental pyramid. These I shall call *the components of the r-system.* The ratios of these components to the corresponding fundamental r-vectors I shall call *the coordinates of the r-system.* They are, of course, all independent, since, if we vary one of them, we vary the system. The number of them is the number of the S_r faces; conversely every $\dfrac{\overline{n+r}!}{\overline{n-r}!\,\overline{r+1}!}$ magnitudes can be regarded as the coordinates of an r-system.

The necessary and sufficient conditions that a system of r-vectors should balance are evidently that its coordinates should all be zero. This statement may be put into the following form:—The moment of the system about every S_{n-r-1} face of the fundamental $(n+1)$-pyramid must vanish. In this form it is a generalization of a well known result in statics.

11. *To find the conditions that an r-System should reduce to a single r-Vector.*

We shall first see that certain conditions are necessary, then that they are sufficient.

If a given r-system be equivalent to a single r-vector, then, by what has been proved, the component r-vectors in the S_r faces through any S_{r-2} of the fundamental pyramid are equivalent to a single r-vector. The S_r's in question are obtained by joining up the $(r-1)$-pyramid of the chosen S_{r-2} to all the edges (S_1's) of the $(n-r+2)$-pyramid of the opposite face. The component r-vectors in question are then represented by numerical multiples of the $(r+1)$-pyramids so obtained, the multipliers being the corresponding coordinates of the r-system. It follows from the section theorem that the same multiples of these edges are likewise equivalent to a

* In the case of a point-vector the above definition will still be valid, if we assign to the fundamental point vectors unit magnitude.

single 1-vector, and we have to express the necessary conditions for this. The conditions have been already obtained in a previous paper,* viz., they are that the 1-vectors in every tetrahedron of the $(n-r+2)$-pyramid must be equivalent to a single 1-vector. In other words: *Take any tetrahedron of the fundamental $(n+1)$-pyramid, and any S_{r-2} face of the opposite S_{n-4}; then, if an r-system be equivalent to a single vector, the components of the system in the S_r faces which pass through the S_{r-2} and the several edges of the tetrahedron are equivalent to a single r-vector.*

These conditions are then necessary; we proceed to show that they are sufficient.

We note first that they entail as consequence that the r-vectors through every S_{r-2} base are equivalent to a single r-vector. This follows, by reasoning similar to the above, from the fact that the corresponding conditions for 1-vectors are sufficient as well as necessary, as was shown in the paper quoted.

Assume that the conditions are sufficient for $(r-1)$-systems and r-systems in S_{n-1}. Take an r-system in S_n satisfying the conditions. Consider the r-vectors of the system which lie in an S_{n-1} face of the fundamental pyramid. In this S_{n-1} take any S_{r-2} face, and any S_3 face not intersecting it. The corresponding r-vectors are then equivalent to a single r-vector. Hence, by hypothesis, all the r-vectors, lying in this S_{n-1} face are equivalent to a single r-vector. The remaining r-vectors all pass through the opposite vertex V. In the S_{n-1} take any S_{r-3} face, and any S_3 face not intersecting it, and consider all the r-vectors which pass through the vertex V, the S_{r-3}, and the several edges of the tetrahedron of the S_3 face. These are equivalent to a single vector. Therefore, by the section theorem, the same is true for the corresponding $(r-1)$-vectors in the S_{n-1}, found by omitting the vertex V. By hypothesis, it follows that all such $(r-1)$-vectors are equivalent to a single vector. Hence, by the section theorem, the same is true for all the original r-vectors through V.

Having now reduced our system to two vectors, the first in an S_{n-1} face, the second through the opposite vertex, it remains to show that these two compound; *i.e.*, that their spaces intersect in an S_{r-1}.

Consider any S_{r-2} face of the S_{n-1} face, and the opposite S_{n-r+1} face of the fundamental $(n+1)$-pyramid. The components of the two vectors through the S_{r-2} are equivalent to a single r-vector.

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* See note *supra*, p. 33.

But they will meet the S_{n-r+1} in two straight lines, one of which passes through V, the other lies in the S_{n-1} face. By the section theorem these lines must intersect, the point of intersection lying, of course in the S_{n-r} opposite to the S_{r-2} in the S_{n-1} face. Now these lines are known, by § 9, to lie in the S_r's of the original r-vectors. Therefore the S_r's of the two r-vectors intersect in a point of the S_{n-r} in question.

Similarly the S_r's of the two vectors intersect in a point of every S_{n-r} face of the S_{n-1}. Therefore, by a property* of an n-pyramid, they have in common S_{r-1}. The two vectors therefore compound into a single vector, as was to be proved.

Here we assumed that the conditions were sufficient for $(r-1)$-vectors, and for r-vectors in S_{n-1}. Now they are so for point-vectors and for line-vectors in S_2. Therefore they are so universally.

It will be noted that, since two S_{n-1}'s always intersect in an S_{n-2}, an $(n-1)$-system, like a point-system, is *always* reducible to a single vector. Inspection of the argument will show that it breaks down in this case.

12. *Flat-Space Coordinates.*

We may define the coordinates of an S_r to be the coordinates of a unit vector in that S_r. In this case there is, of course, an inhomogeneous equation between the coordinates, expressing the fact that the resultant is of unit magnitude. If we prefer to avoid this equation, we may concern ourselves with relative values only, and define the coordinates of an S_r to be the ratios of the coordinates of any r-vector in that S_r. It at once follows from the preceding article that the coordinates of an S_r satisfy equations of which the following is a type :—

$$p_{1256\ldots r+3}\,p_{3456\ldots r+3}+p_{2356\ldots r+3}\,p_{1456\ldots r+3}+p_{3156\ldots r+3}\,p_{2456\ldots r+3}=0,$$

where, with the usual notation, the coordinates of an S_r are denoted by the letter p with $(r+1)$-indices chosen from the numbers $1, 2, \ldots, n+1$.

The equations are easily seen to form a complete system. For it follows at once from the preceding article that there are no others which are not consequences of these. Moreover they themselves are linearly independent, since we can satisfy all of them but one by

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* This follows from the fact that an S_p cannot meet every S_q face of a fundamental n-pyramid in an S_{n-1}, unless it meets every S_q of the S_{n-1}; that is, unless $p+q \geqq n-1$. Thus, if two spaces intersect in a point at least of every S_{n-r} face, their intersection must be an S_{r-1} at least.

taking an r-system consisting of two r-vectors in S_r faces intersecting in an S_{r-2} face.

In the special case of point-vectors ($r=0$), it will be at once recognized that the coordinates here defined are the same as those invented by Möbius.

13. THEOREM.—*The component in any face of the moment of one vector about another is the moment of the component of the first about that of the second.*

It is only necessary to prove this for an S_{r-1} face; the result then follows by repetition of the process. That the theorem is true as regards position is clear from the rule of § 7. To prove it as regards magnitude choose all the vertices but one of the representative pyramid of each vector to lie in the S_{n-1}. Let P, Q be the remaining vertices, and let their projections on the S_{n-1} from the opposite vertex V be P', Q'. Then

$$\frac{\text{component of first vector}}{\text{projection of first vector}} = \frac{VP}{VP'},$$

with a similar equation for the second vector. Therefore

$$\frac{\text{moment of components}}{\text{moment of projections}} = \frac{VP \,.\, VQ}{VP' \,.\, VQ'} = \frac{\text{area of triangle } VPQ}{\text{area of triangle } VP'Q'}.$$

But, by the rule given at the end of § 7, this is the same as

$$\frac{\text{component of moment}}{\text{projection of moment}}.$$

But the denominators of the first and last fractions are evidently the same; hence the numerators are equal, which proves the theorem.

14. *Given the coordinates of an S_p and an S_q which do not intersect, to find those of the S_{p+q+1} containing them.*

Take a unit vector in each of the given flat-spaces. Then, by the last article, the component of the moment in any S_{p+q+1} face of the fundamental pyramid is the moment of the components; that is, is the sum of the moments of all the component p- and q-vectors in S_p and S_q faces of the S_{p+q+1} face. Since every such S_p intersects every such S_q except one, viz., the opposite S_q of the S_{p+q+1} face, each such S_p will contribute only one term to the sum. This term is the product of the coordinate of the p-vector with respect to the S_p face, and that of the q-vector with respect to the opposite S_q face,

multiplied by the volume of the $(p+q+2)$-pyramid of the S_{p+q+1} face.

Hence :—

The coordinate with respect to this S_{p+q+1} face of the moment of the unit p-vector about the unit q-vector is equal to the sum of all possible products of coordinates of those vectors in opposite S_p and S_q faces of this S_{p+q+1} face.

The coordinates of the S_{p+q+1} are therefore given by the ratios of these products.

15. *Given an S_p and an S_q (where $q = n-p+r$), intersecting in an S_r and no more, to find the coordinates of the S_r.*

We shall require the following lemma :—

Given a p-vector, arrange that all but $(r+1)$ of the vertices of the representative $(p+1)$-pyramid lie in a particular S_{n-r-1} face. The pyramid obtained by taking these $(p-r)$ vertices with the $(r+1)$ vertices of the fundamental pyramid lying in the opposite S_r face defines a p-vector, the ratio to which of the component of the original p-vector through this S_r face is the same for all vectors whose representative pyramids have the $(r+1)$-points in common, and is equal to that coordinate of the r-vector, determined by the $(r+1)$-points, which has reference to the S_r face in question.

The truth of this lemma is obvious, since, obtaining the component in question by repeated use of the construction of § 7, the $(p-r)$-points are left unaffected, and their number and position might be altered without affecting the argument. In particular, if they are all absent and we are only dealing with the r-vector determined by the $(r+1)$-points, the ratio is by definition the corresponding coordinate of the r-vector.

The common S_r cannot intersect all the S_{n-r-1} faces, and in general will intersect none of them; consider first one which it does not intersect. This S_{n-r-1} intersects the S_p in an S_{p-r-1}, of which every point is distinct from those of the S_r; $(p-r)$ of the vertices of the representative unit $(p+1)$-pyramid may be chosen in the S_{p-r-1}. These points determine a $(p-r-1)$-vector, whose coordinates we proceed to determine. Such of them as do not vanish, are, by the section theorem, the same as the non-vanishing coordinates of the p-vector determined by the same $(p-r)$-points together with the $(r+1)$-vertices of the fundamental pyramid lying in the S_r

opposite to the chosen S_{n-r-1}. But, by the lemma, the non-vanishing coordinates of this last p-vector are $\frac{1}{x}$ times the corresponding coordinates of the original p-vector, where x is the coordinate of the r-vector in the common S_r, with respect to the S_r face in question. Hence the non-vanishing coordinates of the $(p-r-1)$-vector (*i.e.*, those which refer to S_{p-r-1} faces in the S_{n-r-1} face chosen) are $\frac{1}{x}$ times those coordinates of the original p-vector in S_p's which refer to the S_p faces through the S_r opposite to the S_{n-r-1} face.

We can now, by the theory of moments in § 5, write down the moment of the $(p-r-1)$-vector about the q-vector. If this be M, and Δ the volume of the fundamental pyramid,

$\frac{M}{\Delta}$ = sum of products of coordinates of $(p-r-1)$-vector and q-vector in opposite faces of the fundamental pyramid.

Hence

$\frac{M}{\Delta}x$ = sum of products of coordinates of p-vector with respect to S_p faces through the chosen S_r face, and of coordinates of q-vector with respect to S_q faces through the same S_r face, these being taken together so that the S_p and S_q of the coordinates in any one product intersect in that S_r only.

x being the coordinate of the r-vector, the corresponding coordinate of the r-space is also proportional to the right-hand side of the above equation.

16. *Expression for Flat-Space Coordinates.*

The two preceding theorems enable us to express the coordinates of an S_p in terms of $(p+1)$ S_0's in it, or of $(n-p)$ S_{n-1}'s through it. It is evident that the expressions so obtained are the well known determinantal ones for the coordinates of an S_p in terms of point coordinates and of the coordinates dual to point coordinates. This shows that the coordinates of an S_p as defined in the present paper are identical in fact with those usually given analytically in terms of point coordinates, provided we interpret these point coordinates as those of Möbius.

The theorems in question give us, of course, many other modes of representing the coordinates of an S_p in terms of those of other

spaces. Such a mode of representation arises, for example, whenever $(p+1)$ or $(n-p)$ is not a prime; viz., *if*

$$p+1 = ab,$$

we can express the coordinates of an S_p in terms of those of a S_{b-1}'s or of b S_{a-1}'s lying in it.

Further, *if* $$n-p = ab,$$

the coordinates of an S_p can be expressed in terms of those of a S_{n-b}'s, or of b S_{n-a}'s passing through it.

It is, for instance, easily seen, by continued application of § 14, that the coordinates of an S_{2k-1}, in terms of the coordinates of k straight lines in it, are given by the following rule:—

Take a 1-system formed by unit forces along the k given lines. In any S_{2k-1} face take k non-intersecting edges, multiply together the corresponding coordinates of the 1-system, and sum for all such sets of edges in the S_{2k-1} face: the result is proportional to the corresponding coordinate of the S_{2k-1} sought.

If, instead of unit forces, we choose to take forces of any convenient magnitude, the rule will, by reason of the quadratic equations connecting line coordinates, be unaffected.

17. *The Principle of Duality.*

Any k quantities, where $k = \dfrac{m+1!}{r+1!\, n-r!}$, which satisfy the quadratic equations of S_r coordinates may be taken as proportional to the coordinates of an S_r. Now k, as well as the quadratic equations, is symmetrical in r and $(n-1-r)$; the same ratios may therefore be taken as the coordinates of an S_{n-r-1}. For a coordinate $p_{12\dots r}$, we merely have to write the coordinate $p_{r+1\dots n+1}$. The two spaces may be said to be dual to one another with respect to the fundamental pyramid. It is easy to see that they never intersect. In fact the moment of the one about the other is expressible as the sum of squares of coordinates of one of them. Previous work shows that, if two S_r's intersect in an S_k, their dual S_{n-r-1}'s lie in an S_{n-k-1}; and any laws obeyed by an S_r are easily seen to be true, *mutatis mutandis*, of its dual space.

18. *Canonical Form of an r-System.*

The theory of the composition and resolution of r-vectors has an *interest of its own* independent of its application to S_r-geometry. It

would, perhaps, be an admissible extension of the term to speak of an S_r-statics. Some of the principal results when r is equal to 1 have been given by me in a paper laid before The London Mathematical Society in March of this year (1898).* A new theory will present itself first for S_2's in S_5, the fundamental problem being to find the canonical form for a system of 2-vectors.

The case when r is equal to $n-2$ has an important application to the theory of motion of rigid bodies in space of n dimensions. By the principle of duality, explained in the preceding article, it follows that the laws of composition and resolution of $(n-2)$-vectors are the same as those for 1-vectors. Making use of the results of the paper above referred to, we at once obtain the following theorem :—

Any system of $(n-2)$-vectors in S_n, where n is even, is equivalent to a system whose constituent vectors all pass through a fixed point, or more generally a fixed S_{2k}. If n be odd, no such specialization occurs in general; but, if the given system can be reduced to one whose constituent vectors all pass through an S_{2k}, they will all pass through an S_{2k+1}.

Thus we may say:

The general instantaneous motion of a solid in space of an even number of dimensions is such that a point, or a plane, or an S_4, &c., remains at rest. In space of an odd number of dimensions, the motion is unspecialized in general; but, if any S_{2k} be at rest, then all the points of an S_{2k+1} are at rest.†

The case when r is equal to 2 and all the vectors concerned have a common point corresponds to the theory of couples in space of n dimensions. The section theorem shows that the theory of such 2-vectors is the same as that of 1-vectors in space of $(n-1)$ dimensions. Thus:

Any system of couples S_n, when n is even, reduce in general to $\frac{n}{2}$ couples, and, when n is odd, to $\frac{n-1}{2}$ couples.

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* *Proc.*, Vol. XXIX., pp. 478–487.

† *Cf.* Clifford, "On the Classification of Loci," *Phil. Trans. Roy. Soc.*, Pt. II., 1878, p. 669; *Math. Papers*, 305–331.

The Structure of certain Linear Groups with Quadratic Invariants. By L. E. Dickson, Ph.D. Received September 1st, 1898. Communicated by Prof. E. H. Moore, November 10th, 1898.

1. Consider the group $L_{m,n,p}$ of all linear substitutions

$$(1)\quad \left\{\begin{aligned} \xi_i' &= \sum_{j=1}^{m} (a_{ij}\xi_j + \gamma_{ij}\eta_j) \\ \eta_i' &= \sum_{j=1}^{m} (\beta_{ij}\xi_j + \delta_{ij}\eta_j) \end{aligned}\right\} \quad (i = 1, \ldots, m)$$

in the $GF[p^n]$* which have the absolute invariant

$$\xi_1\eta_1 + \xi_2\eta_2 + \ldots + \xi_m\eta_m.$$

The conditions thus imposed upon the coefficients are

$$(2)\quad \left\{\begin{array}{ll} \sum_{i=1}^{m} a_{ij}\beta_{ij} = 0, & \sum_{i=1}^{m} \gamma_{ij}\delta_{ij} = 0 \\ \sum_{i=1}^{m} (a_{ij}\beta_{ik} + a_{ik}\beta_{ij}) = 0, & \sum_{i=1}^{m} (\gamma_{ij}\delta_{ik} + \gamma_{ik}\delta_{ij}) = 0 \\ \sum_{i=1}^{m} (a_{ij}\delta_{ik} + \beta_{ij}\gamma_{ik}) = 0, & \sum_{i=1}^{m} (a_{ij}\delta_{ij} + \beta_{ij}\gamma_{ij}) = 1 \end{array}\right\}$$

$$(j, k = 1, \ldots, m, j \neq k).$$

The inverse of the substitution (1) of the group $L_{m,n,p}$ is

$$(1)_{-1}\quad \left\{\begin{aligned} \xi_i' &= \sum_{j=1}^{m} (\delta_{ji}\xi_j + \gamma_{ji}\eta_j) \\ \eta_i' &= \sum_{j=1}^{m} (\beta_{ji}\xi_j + a_{ji}\eta_j) \end{aligned}\right\} \quad (i = 1, \ldots, m).$$

Indeed, the product (1) $(1)_{-1}$ replaces ξ_k by

$$\sum_{i=1}^{m} \left\{ \sum_{j=1}^{m} (a_{ji}\delta_{jk} + \beta_{ji}\gamma_{jk})\,\xi_i + \sum_{j=1}^{m} (\gamma_{ji}\delta_{jk} + \gamma_{jk}\delta_{ji})\,\eta_i \right\} \equiv \xi_k.$$

Further, (1) and its inverse have the same determinant ± 1. In fact, if we reflect on the main diagonal the determinant of $(1)_{-1}$, and then

[* For the abstract theory of the Galois field of order p^n, denoted by the symbol $GF[p^n]$, we refer to the paper by Prof. Moore in the *Congress Mathematical Papers* of 1893.]

interchange the first and second, the third and fourth, &c., rows as well as columns, we obtain the determinant of the substitution (1).

Denote by $(2)_{-1}$ the relations obtained from (2) [and equivalent to them] by replacing $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij}$, by $\delta_{ji}, \beta_{ji}, \gamma_{ji}, \alpha_{ji}$ respectively.

Among the substitutions (1) occur the following (where only the indices altered are written),

$$W_{i,j,\lambda}: \ \xi_i' = \xi_i + \lambda\eta_j, \quad \xi_j' = \xi_j - \lambda\eta_i;$$
$$V_{i,j,\lambda}: \ \eta_i' = \eta_i + \lambda\xi_j, \quad \eta_j' = \eta_j - \lambda\xi_i;$$
$$Q_{i,j,\lambda}: \ \xi_i' = \xi_i + \lambda\xi_j, \quad \eta_j' = \eta_j - \lambda\eta_i;$$
$$T_{i,\lambda}: \ \xi_i' = \lambda\xi_i, \qquad \eta_i' = \lambda^{-1}\eta_i;$$
$$P_{ij} = (\xi_i\xi_j)(\eta_i\eta_j); \quad S_{ij} = (\xi_i\eta_i)(\xi_j\eta_j).$$

2. THEOREM.—*The substitutions of the above types generate a sub-group $L'_{m,n,p}$ of index 2 under $L_{m,n,p}$, the former being extended to the latter by the substitution $(\xi_i\eta_i)$. $L'_{m,n,p}$ contains, for $p > 2$, all the substitutions of $L_{m,n,p}$ of determinant $+1$; for $p = 2$, all satisfying* the relation in the $GF[2^n]$,*

$$\sum_{i,j}^{1\ldots m} \alpha_{ij}\delta_{ij} = m.$$

Let Σ be an arbitrary substitution (1) of $L_{m,n,p}$. We first find a substitution S derived from the above $W_{i,j,\lambda}$, &c., which (like Σ) replaces ξ_1 by

$$f_1 \equiv \sum_{j=1}^{m} (\alpha_{1j}\xi_j + \gamma_{1j}\eta_j),$$

where, by $(2)_{-1}$,
$$\sum_{j=1}^{m} \alpha_{1j}\gamma_{1j} = 0.$$

(*a*) If $\alpha_{11} \neq 0$, we may take for S the product

$$T_{1,\alpha_{11}} Q_{1,2,\alpha_{12}} W_{1,2,\gamma_{12}} \cdots Q_{1,m,\alpha_{1m}} W_{1,m,\gamma_{1m}}.$$

(*b*) If $\alpha_{11} = 0$, $\gamma_{11} \neq 0$, we may choose for S

$$S_{12} T_{1,\gamma_{11}} Q_{1,2,\gamma_{12}} W_{1,2,\alpha_{12}} Q_{1,3,\alpha_{13}} W_{1,3,\gamma_{13}} \cdots Q_{1,m,\alpha_{1m}} W_{1,m,\gamma_{1m}}.$$

(*c*) If $\alpha_{1j} = \gamma_{1j} = 0 \ (j = 1 \ldots s-1)$, while α_{1s} and γ_{1s} are not both zero, we obtain by (*a*) or (*b*) a substitution S' replacing ξ_s by f_1. Then will $S = S'P_{1s}$.

* *Bulletin of the American Mathematical Society*, July, 1898.

We may therefore set $\Sigma = S\Sigma'$, Σ' being a new substitution of $L_{m,n,p}$ which leaves ξ_1 fixed. Let it replace η_1 by

$$f_1' \equiv \sum_{j=1}^{m} (\beta_{1j}\xi_j + \delta_{1j}\eta_j),$$

where, by $(2)_{-1}$, we have

$$\delta_{11} = 1, \quad \beta_{11} + \beta_{12}\delta_{12} + \ldots + \beta_{1m}\delta_{1m} = 0.$$

Hence $$S_1 \equiv V_{2,1,-\beta_{12}} Q_{2,1,-\delta_{12}} \ldots V_{m,1,-\beta_{1m}} Q_{m,1,-\delta_{1m}}$$

replaces η_1 by f_1'. We may therefore set $\Sigma' = S'\Sigma_1$, Σ_1 being a new substitution of $L_{m,n,p}$ which leaves ξ_1 and η_1 fixed. For Σ_1, the relations $(2)_{-1}$ give

$$a_{j1} = \gamma_{j1} = \beta_{j1} = \delta_{j1} = 0 \quad (j = 2, \ldots, m),$$

together with relations connecting the a_{ij}, β_{ij}, γ_{ij}, δ_{ij} $(i, j = 2, \ldots, m)$ of precisely the form $(2)_{-1}$ when written for $m-1$ couples of indices. Proceeding with Σ_1 as we did with Σ, we reach finally a substitution Σ_{m-1}, affecting only ξ_m, η_m, and hence either

$$T_{m,\,a_{mm}} \quad \text{or} \quad (\xi_m\,\eta_m)\,T_{m,\,\gamma_{mm}}.$$

Therefore $$\Sigma = L' \quad \text{or} \quad L'(\xi_m\eta_m),$$

where L' is derived from the substitutions tabulated at the end of § 1.

Corollary.—*For $m > 2$, $L'_{m,n,p}$ may be generated by the S_{ij}, $W_{i,j,\lambda}$, together with $T_{1,\nu}$, where ν is some not-square in the $GF[p^n]$.*

This follows from the formulæ of composition

$$Q_{i,i,\lambda} = S_{jk}^{-1} W_{i,j,\lambda} S_{jk}, \quad V_{i,j,\lambda} = S_{ik}^{-1} Q_{i,j,\lambda} S_{ik} \quad (k \neq i \text{ or } j)\,;$$

$$P_{i,j}\,T_{j,-1} = Q_{j,i,-1}\,Q_{i,j,1}\,Q_{j,i,-1}\,;$$

$$T_{1,\mu} T_{2,\mu} = S_{12} P_{12} T_{1,-1} V_{1,2,\mu^{-1}} W_{1,2,\mu} V_{1,2,\mu^{-1}}\,;$$

$$T_{1,\mu^2} = T_{1,\mu} T_{2,\mu} \,.\, T_{2,\mu^{-1}} T_{3,\mu^{-1}} \,.\, T_{3,\mu} T_{1,\mu}.$$

3. The order $O'_{m,n,p}$ of the group $L'_{m,n,p}$ is now readily determined. The linear function f_1 may take $P_{m,n}-1$ values, where $P_{m,n}$ denotes the number of solutions in the $GF[p^n]$ of

$$a_{11}\gamma_{11} + a_{12}\gamma_{12} + \ldots + a_{1m}\gamma_{1m} = 0.$$

By considering the cases $a_{11}\gamma_{11} = 0$ and $a_{11}\gamma_{11} \neq 0$, we derive the recursion formula

$$P_{m,n} = (p^n - 1)\,p^{n(2m-2)} + p^n P_{m-1,n}.$$

Since
$$P_{1,n} = 2p^n - 1,$$
we find by induction that
$$P_{s,n} - 1 = (p^{ns} - 1)(p^{n(s-1)} + 1).$$

Further, the number of values of f_1' is $p^{n(2m-2)}$. Hence
$$O'_{m,n,p} = p^{n(2m-2)}(P_{m,n} - 1)\, O'_{m-1,n,p},$$
with the initial value
$$O'_{1,n,p} = p^n - 1 = \tfrac{1}{2}(P_{1,n} - 1).$$
Hence the order $O'_{m,n,p}$ equals
$$(p^{nm}-1)(p^{n(2m-2)}-1)\,p^{n(2m-2)}\,(p^{n(2m-4)}-1)\,p^{n(2m-4)} \dots (p^{2n}-1)\,p^{2n}.$$

4. Consider, for $m > 2$, the following sub-group of $L'_{m,n,p}$,
$$L''_{m,n,p} \equiv \{W_{i,j,\lambda},\ S_{ij}\ \ (i, j = 1, \dots, m,\ i \neq j)\},$$
λ taking every value in the $GF[p^n]$. It contains every $Q_{i,j,\lambda}$, $V_{i,j,\lambda}$, $P_{i,j}\,T_{j,-1}$, $T_{1,\lambda}\,T_{2,\lambda}$, T_{1,λ^2}.

Let I be an invariant sub-group of $L''_{m,n,p}$ containing a substitution S, given by (1), neither the identity nor $T_{1,-1}\,T_{2,-1} \dots T_{m,-1}$, which* changes the sign of every index.

5. Lemma I. — *I contains a substitution, not the identity, which multiplies ξ_1 by a constant.*

If $f_1 \equiv \sum_{j=1}^{m} (a_{1j}\xi_j + \gamma_{1j}\eta_j)$ does not reduce to $a_{11}\xi_1$, we have the following two cases.

(*a*) $\gamma_{11} \neq 0$.—Then $L''_{m,n,p}$ contains the product T,
$$T_{1,\gamma_{11}^{-1}}\, T_{2,\gamma_{11}^{-1}}\, V_{2,1,-\gamma_{11}a_{12}}\, Q_{2,1,-\gamma_{11}^{-1}\gamma_{12}}\, V_{3,1,-a_{13}}\, Q_{3,1,-\gamma_{13}} \cdots$$
$$\cdots V_{m,1,-a_{1m}}\, Q_{m,1,-\gamma_{1m}},$$
which replaces η_1 by f_1 and ξ_1 by $\gamma_{11}^{-1}\xi_1$. Hence I contains $S_1 = T^{-1}ST$, which replaces ξ_1 by $\gamma_{11}^{-1}\eta_1$. If S_1 multiplies ξ_2 by a constant, I contains its transformed by $P_{12}T_{1,-1}$, which will multiply ξ_1 by the same constant. In the contrary case there exists a substitution T_1 in L'',

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* It may be shown that it is the only substitution of L'' commutative with every substitution in L''.

leaving ξ_1 and η_1 fixed, and not commutative with S_1. For, if we equate the expressions by which $S_1 V_{2,3,\lambda}$ and $V_{2,3,\lambda} S_1$ replace η_3, we get

$$\eta_3' - \lambda \xi_2' = \eta_3' + (\ )\,\xi_2 + (\ )\,\xi_3.$$

Similarly, if S_1 be commutative with $Q_{3,2,\lambda}$, we find

$$\xi_3' + \lambda \xi_2' = \xi_3' + (\ )\,\xi_2 + (\ )\,\eta_3.$$

Hence $\xi_2' = (\ )\,\xi_2$, contrary to hypothesis. The group I therefore contains $S_1^{-1} T_1^{-1} S_1 T_1 \neq 1$, which leaves ξ_1 fixed.

(*b*) $\gamma_{11} = 0$.—Suppose,* for example, that $\alpha_{12} \neq 0$. Then L'' contains the product T,

$$T_{3,\,\alpha_{12}} T_{2,\,\alpha_{12}} Q_{2,\,1,\,\alpha_{11}} Q_{2,\,3,\,\alpha_{12}^{-1}\alpha_{13}} W_{2,\,3,\,\gamma_{13}\alpha_{12}} Q_{2,\,4,\,\alpha_{14}} W_{2,\,4,\,\gamma_{14}} \dots$$
$$\dots Q_{2,\,m,\,\alpha_{1m}} W_{2,\,m,\,\gamma_{1m}},$$

which replaces ξ_2 by f_1 without altering ξ_1. Then I contains S_1, the transformed of S by T, which replaces ξ_1 by ξ_2.

If S_1 multiplies ξ_3 by a constant, its transformed by $P_{13} T_{1,-1}$, multiplies ξ_1 by the same constant. In the contrary case, S_1 is not commutative with both $V_{3,1,\lambda}$ and $V_{3,2,\lambda}$, which leave ξ_1 and ξ_2 fixed. Indeed, $S_1 V_{3,k,\lambda}$ and $V_{3,k,\lambda} S_1$ replace η_k by respectively

$$\eta_k' - \lambda \xi_3', \quad \eta_k' + (\ )\,\xi_3 + (\ )\,\xi_k.$$

Hence would $\xi_3' = \rho \xi_3$. It follows that I contains

$$S_1^{-1} V_{3,k,\lambda}^{-1} S_1 V_{3,k,\lambda} \quad (k = 1, 2),$$

which leave ξ_1 fixed, and do not both reduce to the identity.

NOTE.—It is clear from the proof that the final substitution multiplying ξ_1 by a constant does not reduce to $T_{1,-1} T_{2,-1} \dots T_{m,-1}$, unless S reduces to the same product.

6. LEMMA II.—*The group I contains a substitution, not the identity, leaving fixed ξ_1 and η_1.*

By Lemma I., the group I contains a substitution S, which replaces ξ_1 by $\alpha\xi_1$ and η_1 by

$$\beta_{11}\xi_1 + \alpha^{-1}\eta_1 + \sum_{j=2}^{m} (\beta_{1j}\xi_j + \delta_{1j}\eta_j).$$

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* If $\alpha_{12} = \alpha_{13} = \dots = \alpha_{1m} = 0$, not every $\gamma_{1j} = 0$ by hypothesis. If $\gamma_{12} \neq 0$, we take in place of S its transformed by S_{23}, for which $\alpha_{12}' \neq 0$.

(*a*) $\beta_{11}=0$, $\beta_{1j}=\delta_{1j}=0$ $(j=2,\ldots,m)$.—Then $S=T_{1,a}S_1$, where, by the relations (2), S_1 involves only the indices ξ_i, η_i $(i=2,\ldots,m)$. Thus, if $a=+1$, S leaves ξ_1 and η_1 fixed. Suppose, then, that $a\neq 1$.

If $S_1=T_{2,-1}T_{3,-1}\ldots T_{m,-1}$, the value $a=-1$ is excluded by the hypothesis on S. Since $S_1^2=1$, we have

$$S^2=T_{1,a^2}\neq 1.$$

Transforming it by $P_{1,2}T_{2,-1}$, we obtain in I a substitution leaving ξ_1 and η_1 fixed.

If S_1 does not reduce to the above product, there exists a substitution Σ_1 in the group $L''_{m,n,p}$ affecting the same indices as does S_1, and not commutative with S_1. Then will I contain

$$S^{-1}\Sigma_1^{-1}S\Sigma_1\equiv S_1^{-1}\Sigma_1^{-1}S_1\Sigma_1\neq 1,$$

which leaves ξ_1 and η_1 fixed.

(*b*) $\beta_{11}=0$, β_{1j}, δ_{1j} not all zero.—Then, by § 2, $L''_{m,n,p}$ contains a substitution T, leaving ξ_1 and η_1 fixed, and replacing ξ_2 by

$$\sum_{j=2}^{m}(\beta_{1j}\xi_j+\delta_{1j}\eta_j).$$

Therefore I contains S_1, the transformed of S by T, which replaces ξ_1 by $a\xi_1$, η_1 by $a^{-1}\eta_1+\xi_2$.

Consider the substitutions Σ of L'', such that $S_1^{-1}\Sigma^{-1}S_1\Sigma$ leaves ξ_1 and η_1 fixed. If one such product be not the identity, the Lemma is proven. We may therefore suppose that it is the identity for every suitable Σ, *e.g.*, $Q_{3,2,\lambda}$, $V_{2,3,\lambda}$, T_{3,μ^2}, $T_{1,\lambda}T_{2,\lambda^{-1}}$. Equating the two values by which $S_1V_{2,3,\lambda}$ and $V_{2,3,\lambda}S_1$ replace η_2, and the two values by which they replace η_3, we get

$$\xi_3'=\delta_{22}\xi_3-\delta_{23}\xi_2,\quad \xi_2'=-\delta_{32}\xi_3+\delta_{33}\xi_2.$$

Equating the values by which $S_1Q_{3,2,\lambda}$ and $Q_{3,2,\lambda}S_1$ replace ξ_3, and those by which they replace η_2, we find

$$\xi_2'=a_{33}\xi_2-\gamma_{32}\eta_3,\quad \eta_3'=-\beta_{23}\xi_2+\delta_{22}\eta_3.$$

For $m>3$, we may suppose S_1 to be commutative with $V_{3,4,\lambda}$, so that we have

$$\xi_3'=\mu\xi_3,\quad \eta_3'=\mu^{-1}\eta_3.$$

Transforming S_1 by $P_{13}T_{3,-1}$, we are led to Case (*a*).

For $m = 3$, S_1 becomes, on applying (2),

$$S_1 : \begin{cases} \xi_1' = a\xi_1, & \eta_1' = a^{-1}\eta_1 + \xi_2, \\ \xi_2' = \pm \xi_2, & \eta_2' = \mp a\xi_1 \mp \beta_{23}\delta_{23}\xi_2 \pm \eta_2 + \beta_{23}\xi_3 + \delta_{23}\eta_3, \\ \xi_3' = \pm \xi_3 - \delta_{23}\xi_2, & \eta_3' = \pm \eta_3 - \beta_{23}\xi_2. \end{cases}$$

If $p^n > 2$, two distinct quantities $\lambda \neq 0$ exist in the $GF[p^n]$. Hence, if we equate the values by which $T_{1,\lambda}T_{2,\lambda^{-1}}S_1$ and $S_1T_{1,\lambda}T_{2,\lambda^{-1}}$ replace η_2, we find that $\beta_{23} = \delta_{23} = 0$. We are thus led to Case (a).

Consider the case $p^n = 2$. We may suppose that $\beta_{23}\delta_{23} = 0$, since otherwise the transformed of S_1 by $P_{12}T_{2,-1}$ falls under Case (c) below.

If $\beta_{23} = 0$, $\delta_{23} = 1$, $S_1 \equiv Q_{3,2,1}V_{1,2,1}$, and we have

$$S_1V_{1,3,1}^{-1}S_1V_{1,3,1} = V_{1,2,1}.$$

If $\delta_{23} = 0$, $\beta_{23} = 1$, $S_1 = V_{2,3,1}V_{1,2,1}$, and we have

$$S_1Q_{2,3,1}^{-1}S_1Q_{2,3,1} = V_{1,3,1}.$$

(c) $\beta_{11} \neq 0$.

Since
$$\sum_{j=1}^{m} \beta_{1j}\delta_{1j} = 0,$$

we may suppose, for example, that $\beta_{12} \neq 0$, $\delta_{12} \neq 0$. Transforming S by $T_{2,\delta_{12}}T_{3,\delta_{12}}$, we may suppose that $\delta_{12} = 1$. The product

$$Q_{3,2,-\delta_{13}}V_{2,3,\beta_{13}} \cdots Q_{m,2,-\delta_{1m}}V_{2,m,\beta_{1m}}$$

transforms S into S_1, which replaces ξ_1 by $a\xi_1$, and η_1 by

$$\beta_{11}\xi_1 + a^{-1}\eta_1 - a^{-1}\beta_{11}\xi_2 + \eta_2.$$

Denote by μ the coefficient of ξ_2. If, among the substitutions $Q_{3,2,\mu}W_{2,3,1}$, $T_{3,\mu}T_{2,\mu}S_{23}$, &c., of the group L'', which leave $\mu\xi_2 + \eta_2$ invariant, there exist one, say R, which is not commutative with S_1, then I contains

$$S_1^{-1}R^{-1}S_1R \neq 1,$$

which leaves ξ_1 and η_1 fixed. In the contrary case, we find, on equating the values by which $S_1Q_{3,2,\mu}W_{2,3,1}$ and $Q_{3,2,\mu}W_{2,3,1}S_1$ replace the index ξ_2,

$$\eta_3' = (\ )\,\eta_3 - a_{23}\eta_2 + \mu a_{23}\xi_2.$$

By one of the relations $(2)_{-1}$, we have $a_{23} = 0$. Then, if S_1 be commutative with $T_{3,\rho}\, T_{2,\rho}\, S_{23}$,

$$\xi_3' = \pm \xi_3, \quad \eta_3' = \pm \eta_3.$$

We are again led to Case (a).

7. Applying anew the reasoning of §§ 5, 6, we conclude that, if $m \overset{=}{>} 4$, I contains a substitution leaving fixed $\xi_1, \eta_1, \xi_3, \eta_3$; finally, that I contains a substitution leaving fixed

$$\xi_i, \ \eta_i \quad (i = 1, 2, \ldots, m-2).$$

Transforming it by the product of $P_{1,m-1}\, T_{1,-1}$ and $P_{2,m}\, T_{2,-1}$, we obtain a substitution S of the group I which alters only $\xi_1, \eta_1, \xi_2, \eta_2$, and does not reduce to the identity. Applying the relations (2), it is seen to have the form

$$S \left\{ \begin{aligned} \xi_1' &= a_{11}\xi_1 + \gamma_{11}\eta_1 + a_{12}\xi_2 + \gamma_{12}\eta_2, \\ \eta_1' &= \beta_{11}\xi_1 + \delta_{11}\eta_1 + \beta_{12}\xi_2 + \delta_{12}\eta_2, \\ \xi_2' &= a_{21}\xi_1 + \gamma_{21}\eta_1 + a_{22}\xi_2 + \gamma_{22}\eta_2, \\ \eta_2' &= \beta_{21}\xi_1 + \delta_{21}\eta_1 + \beta_{22}\xi_2 + \delta_{22}\eta_2. \end{aligned} \right.$$

8. THEOREM.—*If* $m > 2$, *I contains a substitution* $V_{a,b,\rho}$.

CASE I.—$\gamma_{11} \neq 0$. Transforming S by

$$T_{1,\gamma_{11}^{-1}}\, T_{3,\gamma_{11}^{-1}}\, V_{2,1,-a_{12}}\, Q_{2,1,-\gamma_{12}},$$

we obtain a substitution S_1 in I which replaces ξ_1 by $\gamma_{11}^{-1}\eta_1$, but otherwise of the same form as S. Applying the relations (2), S_1 is seen to take the form

$$\left\{ \begin{matrix} 0 & \gamma_{11}^{-1} & 0 & 0 \\ \gamma_{11} & -\gamma_{11}^{-1}\beta_{12}\delta_{12} & \beta_{12} & \delta_{12} \\ 0 & -\gamma_{11}^{-1}\beta_{12}\gamma_{22} & 0 & \gamma_{22} \\ 0 & -\gamma_{11}^{-1}\delta_{12}\gamma_{22}^{-1} & \gamma_{22}^{-1} & 0 \end{matrix} \right\}.$$

If S_1 be not commutative with S_{23}, I contains

$$S_1^{-1} S_{23} S_1 S_{23} \neq 1,$$

which leaves fixed ξ_1, ξ_3, η_3 (Case III.). But, if S_1 be commutative with S_{23}, the matrix of its coefficients must remain unaltered upon interchanging the last two rows and simultaneously the last two

columns. Hence $\beta_{12} = \delta_{12}, \quad \gamma_{22} = \gamma_{22}^{-1} = \pm 1,$

$$S_1 = T_{1,\gamma_{11}} S_{12} Q_{2,1,-\beta_{12}} V_{1,2,\beta_{12}} T_{2,\pm 1}.$$

Dropping the subscripts to γ_{11} and β_{12}, we consider two cases:

(*a*) $\beta = 0$.—I contains the product of S_1 and its transformed by $P_{12} T_{3,-1}$, viz.,

$$T_{1,\gamma} S_{12} T_{2,\pm 1} \cdot T_{2,\gamma} S_{12} T_{1,\pm 1} = T_{1,\pm\gamma} T_{2,\pm\gamma^{-1}}.$$

Transforming this by S_{23} and $S_{23} V_{1,3,\lambda}$, we get

$$T_{1,\pm\gamma} T_{2,\pm\gamma}, \quad V_{1,3,\lambda(\pm\gamma-1)} T_{1,\pm\gamma} T_{2,\pm\gamma}.$$

Hence, unless $\gamma = \pm 1$, I contains a $V \neq 1$. There remains the case $S_1 = S_{12} T_{1,\pm 1} T_{2,\pm 1}$, which $V_{1,2,\lambda}$ transforms into

$$V_{1,2,-\lambda} W_{1,2,\lambda} S_{12} T_{1,\pm 1} T_{2,\pm 1}.$$

Hence I contains $V_{1,2,-\lambda} W_{1,2,\lambda}$, which replaces ξ_1 by

$$(\lambda^2 + 1)\,\xi_1 + \lambda\eta_2,$$

and consequently falling under Case II.

(*b*) $\beta \neq 0$.—Transforming S_1 by $V_{1,2,\mp\beta}$, we get

$$V_{1,2,\pm\beta} T_{1,\gamma} S_{12} Q_{2,1,-\beta} T_{2,\pm 1}.$$

Its product by S_1^{-1}, which may be written

$$T_{2,\pm 1} Q_{2,1,\beta} S_{12} T_{1,\gamma^{-1}} W_{1,2,-\beta\gamma^{-1}},$$

gives $V_{1,2,\pm\beta} W_{1,2,-\beta\gamma^{-1}}$ (Case II.).

Case II.—$\gamma_{11} = 0$, α_{12} and γ_{12} not both zero. According as $\alpha_{12} \neq 0$ or $\gamma_{12} \neq 0$, the substitution T,

$$T_{2,\alpha_{12}} T_{3,\alpha_{12}} Q_{2,1,\alpha_{12}}, \quad S_{23} T_{2,\gamma_{12}} T_{3,\gamma_{12}} Q_{2,1,\alpha_{11}},$$

replaces ξ_2 by $\alpha_{11}\xi_1 + \gamma_{11}\eta_1 + \alpha_{12}\xi_2 + \gamma_{12}\eta_2$ and leaves ξ_1 fixed. Hence $S_1 \equiv T^{-1} S T$ replaces ξ_1 by ξ_2 and leaves ξ_3, η_3 fixed.

If S_1 be not commutative with $V_{1,2,\lambda}$, I contains

$$S_1^{-1} V_{1,2,\lambda}^{-1} S_1 V_{1,2,\lambda} \neq 1,$$

which leaves ξ_1 fixed (Case III.). But, if S_1 be commutative with $V_{1,2,\lambda}$, it reduces to

$$P_{12} T_{2,-1} Q_{2,1,\delta_{11}} V_{1,2,-\beta_{11}} = \left\{ \begin{matrix} 0 & 0 & 1 & 0 \\ \beta_{11} & \delta_{11} & -\beta_{11}\delta_{11} & 1 \\ -1 & 0 & \delta_{11} & 0 \\ 0 & -1 & \beta_{11} & 0 \end{matrix} \right\}.$$

Then I contains

$$(P_{12}T_{2,-1})^{-1}S_1(P_{12}T_{2,-1})\,S_1^{-1}=Q_{2,1,\delta_{11}}\,Q_{1,2,\delta_{11}}.$$

Its transformed by $T_{1,\rho}T_{2,\rho}$ is $Q_{2,1,\rho^{-1}\delta_{11}}\,Q_{1,2,\rho\delta_{11}}$. Thus I contains the product (where δ is written for δ_{11})

$$Q_{2,1,\rho^{-1}\delta}\,Q_{1,2,\rho\delta}\cdot Q_{2,1,\sigma^{-1}\delta}\,Q_{1,2,\sigma\delta},$$

which, for $\sigma=-\rho\,(1+\delta^2)$, reduces to

$$T_{1,1+\delta^2}\,T_{2,1+\delta^2}^{-1}.$$

Hence, by Case I., (a), I contains a $V\neq 1$, unless $1+\delta^2=0$ or 1. If $1+\delta^2=0$, the cube of $Q_{2,1,\delta}\,Q_{1,2,\delta}$ gives $T_{1,-1}\,T_{2,-1}$, so that for $p>2$ we reach a $V\neq 1$. For $p=2$, we obtain* from $Q_{2,1,1}\,Q_{1,2,1}$ the substitution $Q_{2,1,1}$. If $\delta=0$, we have

$$S_1^2=T_{1,-1}\,T_{2,-1}\,V_{1,2,-2\beta_{11}},\quad S_1^4=V_{1,2,4\beta_{11}}.$$

For the case $p=2$, $Q_{1,2,1}$ transforms $S_1=P_{12}V_{1,2,\beta_{11}}$ into a substitution $\neq 1$, leaving ξ_1 fixed (Case III.).

Case III.—$\gamma_{11}=\alpha_{12}=\gamma_{12}=0$.— We may readily verify that S reduces to

$$T_{1,\alpha_{11}}\,V_{1,2,\beta_{12}}\,Q_{2,1,-\delta_{12}}\,T_{2,\alpha_{22}}.$$

Its transformed by $T_{3,\alpha_{22}}^{-1}\,T_{2,\alpha_{22}}^{-1}\,V_{1,2,\lambda}$ gives

$$V_{1,2,\lambda(\alpha_{11}\alpha_{22}-1)}\,T_{2,\alpha_{22}}\,T_{1,\alpha_{11}}\,V_{1,2,\beta_{12}}\,Q_{2,1,-\delta_{12}}.$$

Hence I contains $V_{1,2,\lambda(\alpha_{11}\alpha_{22}-1)}$.

For the case $\alpha_{22}=\alpha_{11}^{-1}$, I contains

$$\begin{aligned} S' &= S\,(T_{3,-\alpha_{11}}\,T_{2,\alpha_{11}}\,P_{12})^{-1}\,S\,(T_{3,-\alpha_{11}}\,T_{2,\alpha_{11}}\,P_{12}) \\ &= V_{1,2,\beta(\alpha-1)}\,Q_{2,1,-\alpha\delta}\,Q_{1,2,-\delta}, \end{aligned}$$

if we write α for α_{11}, β for β_{12}, δ for δ_{12}.

Transforming S' by S_{13} and the result by $P_{12}T_{2,-1}$, we obtain respectively,

$$Q_{1,2,\beta(\alpha-1)}\,W_{1,2,\alpha\delta}\,V_{1,2,-\delta},$$

$$Q_{2,1,-\beta(\alpha-1)}\,W_{1,2,\alpha\delta}\,V_{1,2,-\delta}.$$

The product of the former by the reciprocal of the latter gives

$$Q_{1,2,\beta(\alpha-1)}\,Q_{2,1,\beta(\alpha-1)}.$$

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* Jordan, *Traité des Substitutions*, p. 204, ll. 15–17.

If $\beta(a-1) \neq 0$, I contains a $V \neq 1$ (end of Case II.). If $a = 1$,

$$S' = Q_{2,1,-\delta}\, Q_{1,2,-\delta}.$$

More generally, for $a_{22} = a_{11}^{-1}$, I contains

$$S(P_{12}T_{3,-1})^{-1} S (P_{12}T_{3,-1}) = Q_{2,1,-\delta_{12}}\, Q_{1,2,-\delta_{12}}.$$

There remains the case $\delta = 0$, $\beta(a-1) = 0$, when S reduces to $T_{1,a}\, T_{2,a^{-1}}$ or $V_{1,2,\beta}$.

9. Having a substitution of the form $V_{a,b,\rho}$, the group I will contain its transformed by $T_{a,\lambda}\, T_{c,\lambda}$, giving $V_{a,b,\rho\lambda^{-1}}$. Hence I contains $V_{a,b,\mu}$ (μ = arbitrary).

Transforming it by $P_{a,i}\, P_{b,j}$, which belongs to L'', we obtain $V_{i,j,\mu}$. This is transformed by S_{ik} and S_{ij} into respectively $Q_{i,j,\mu}$ and $W_{i,j,\mu}$. Further, I contains

$$S_{12} = W_{1,2,-1}\, Q_{1,2,+1}\, V_{1,2,-1}\, Q_{2,1,-1}\, Q_{1,2,+1}\, W_{1,2,-1}.$$

Hence the invariant sub-group I coincides with $L''_{m,n,p}$. We have therefore reached the following result :—

THEOREM.—*For $p > 2$, the maximal invariant sub-group of $L''_{m,n,p}$ is of order 2 or 1, according as L'' contains $T_{1,-1}\, T_{2,-1} \dots T_{m,-1}$ or does not; while, for $p = 2$, $L''_{m,n,p}$ coincides with $L'_{m,n,p}$ and is simple.*

10. The group* $L_{m,n,2}$ with the factors of composition 2 and $O'_{m,n,2}$ (given in § 3) is a direct generalization of Jordan's first hypo-Abelian group G_0.†

It is proved below (§§ 16, 18) that the senary group $L''_{3,n,p}$ ($p > 2$) does not contain a substitution of the form $T_{i,\nu}$, ν being a not-square in the $GF[p^n]$, and it seems probable that a like result holds for the general group $L''_{m,n,p}$. If this conjecture be always true, L'' is of index 2 under L', and L'' is simple only when m is odd, with -1 a not-square in the $GF[p^n]$ ($p > 2$); while, if m be even or if -1 be a square, L'' has the factors of composition 2 and $\frac{1}{4}O'_{m,n,p}$. Independent of the truth of our conjecture, we have the result that, if m be

* The structure of this group was determined by the writer in March, 1897, and published in the *Quarterly Journal*, June, 1898. The present paper uses the simplified treatment given in the article "The Structure of the Hypo-Abelian Groups," *Bulletin of the American Mathematical Society*, July, 1898.

† *Traité des Substitutions*, pp. 199–206.

odd and -1 a not-square, simple groups of order $\frac{1}{2}O'_{m,n,p}$ exist. Indeed, this result follows whether or not $T_{1,-1}$ belongs to $L''_{\pi,n,p}$.

Isomorphism of $L''_{4,n,p}$ with the group $G_{4,n,p}$ of Quaternary Linear Substitutions in the $GF[p^n]$ of determinant unity, §§ 11–19.

11. Consider the composition of quaternary substitutions

$$S: \quad \xi'_i = \sum_{j=0} a_{ij}\xi_j \quad (i = 0, 1, 2, 3).$$

If (a'_{ij}) operates first, we have the composition formula

$$(a''_{ij}) = (a_{ij})(a'_{ij}),$$

where
$$a''_{ij} = \sum_{k=0}^{3} a_{ik}a'_{kj} \quad (i, j = 0, 1, 2, 3).$$

Denoting the second minors of the determinant $|\, a_{ij} \,|$ as follows:—

$$\begin{vmatrix} a_{ij} & a_{il} \\ a_{kj} & a_{kl} \end{vmatrix} \equiv D^{ij}_{kl} \equiv D^{kl}_{ij} \equiv -D^{il}_{kj} \equiv -D^{kj}_{il},$$

we readily verify the formula

$$D''^{ij}_{kl} = \sum_{r<s}^{r,s=0,..,3} D^{ir}_{ks} D'^{rj}_{sl}.$$

12. If, therefore, we build a senary substitution Σ whose thirty-six coefficients are the determinants

$$\pm D^{ij}_{kl} \quad \begin{pmatrix} i<k \\ j<l \end{pmatrix},$$

such that the pair ${}^{i}_{k}$ is constant for the six elements of any row, and the pair ${}^{j}_{l}$ constant for the six elements of any column, and furthermore such that the two elements

$$\pm D^{ij}_{kl}, \quad \pm D^{ji}_{lk}$$

are symmetrically placed with respect to the main diagonal, we obtain a correspondence between the substitutions S and Σ, such that to a given S corresponds one Σ, and to every product $S'S$ corresponds the product $\Sigma'\Sigma$. (An explicit formulation of the proof is given in § 14 for the most interesting case.) The determinant of Σ equals the *second compound** Δ_6 of the determinant $|\, a_{ij} \,|$ of the fourth order. Hence†

$$|\, \Sigma \,| = |\, S \,|^3.$$

* Muir, *Theory of Determinants*, §§ 170–176.
† *Ibid.*, § 174.

13. Of the various isomorphisms thus defined, we proceed to select the one which establishes the isomorphism of the groups $L''_{3,n,p}$, $G_{4,n,p}$.

We begin with the second compound of $|a_{ij}|$, viz.,

$$\Delta_6 \equiv \begin{vmatrix} D_{11}^{00} & D_{12}^{00} & D_{13}^{00} & D_{12}^{01} & D_{13}^{01} & D_{13}^{02} \\ D_{21}^{00} & D_{22}^{00} & D_{23}^{00} & D_{22}^{01} & D_{23}^{01} & D_{23}^{02} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ D_{31}^{20} & D_{32}^{20} & D_{33}^{20} & D_{32}^{21} & D_{33}^{21} & D_{33}^{22} \end{vmatrix}.$$

The value of the determinant is unaltered if we move the first and second rows into the places of the sixth and fourth rows respectively, and the first and second columns into the places of the sixth and fourth columns. The elements of the determinant appear in their new order in the following scheme, where a, b, c take either value ± 1 independently*:

$$\begin{array}{c|cccccc} & \xi_0 & \eta_0 & \xi_1 & \eta_1 & \xi_2 & \eta_2 \\ \hline \xi_0 & 1D_{33}^{00} & aD_{32}^{01} & bD_{33}^{01} & -abD_{32}^{00} & cD_{33}^{02} & acD_{31}^{00} \\ \eta_0 & aD_{23}^{10} & 1D_{22}^{11} & abD_{23}^{11} & -bD_{22}^{10} & acD_{23}^{12} & cD_{21}^{10} \\ \xi_1 & bD_{33}^{10} & abD_{32}^{11} & 1D_{33}^{11} & -aD_{32}^{10} & bcD_{33}^{12} & abcD_{31}^{10} \\ \eta_1 & -abD_{23}^{00} & -bD_{22}^{01} & -aD_{23}^{01} & 1D_{22}^{00} & -abcD_{23}^{02} & -bcD_{21}^{00} \\ \xi_2 & cD_{33}^{20} & acD_{32}^{21} & bcD_{33}^{21} & -abcD_{32}^{20} & 1D_{33}^{22} & aD_{31}^{20} \\ \eta_2 & acD_{13}^{00} & cD_{12}^{01} & abcD_{13}^{01} & -bcD_{12}^{00} & aD_{13}^{02} & 1D_{11}^{00} \end{array}$$

This matrix has the symmetry with respect to the main diagonal and the other properties required for the isomorphism. Further, the square array of signs

$$\begin{array}{cccccc} 1 & a & b & -ab & c & ac \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ ac & c & abc & -bc & a & 1 \end{array}$$

has the property that any row is obtained from the first row by

* It may be verified that for no other arrangement of signs will the substitution leave $\xi_0\eta_0 + \xi_1\eta_1 + \xi_2\eta_2$ invariant.

multiplying the latter by the leading term of the given row. Hence the determinant of the above substitution

$$= 1^2 a^2 b^2 (-ab)^2 c^2 (ac)^2 \Delta_6 \equiv | a_{ij} |^3.$$

14. We find by inspection that the above substitution may be written as follows :—

$$\Sigma : \left\{ \begin{aligned} \xi_i' &= \epsilon_i \sum_j^{0,1,2} (D^{ij}_{33} \epsilon_j \xi_j + a D^{i\,j+1}_{3\,j+2} \epsilon_j \eta_j) \\ \eta' &= \epsilon_i \sum_j^{0,1,2} (a D^{i+1\,j}_{i+2\,3} \epsilon_j \xi_j + D^{i+1\,j+1}_{i+2\,j+2} \epsilon_j \eta_j) \end{aligned} \right\} \quad (i = 0, 1, 2),$$

where the sums $i+1$, $i+2$, $j+1$, $j+2$ are taken modulo 3, and where

$$\epsilon_0 = 1, \quad \epsilon_1 = b, \quad \epsilon_2 = c, \quad \epsilon_i^2 = 1.$$

Introduce the new indices*

$$x_i = \epsilon_i \xi_i, \quad y_i = a \epsilon_i \eta_i \quad (i = 0, 1, 2),$$

so that $$\sum_{i=0}^{2} x_i y_i = a \sum_{i=0}^{2} \xi_i \eta_i.$$

The substitution Σ takes the simple form

$$\Sigma_1 : \left\{ \begin{aligned} x_i' &= \sum_j^{0,1,2} (D^{i\,j}_{3\,3} x_j + D^{i\,j+1}_{3\,j+2} y_j) \\ y_i' &= \sum_j (D^{i+1\,j}_{i+2\,3} x_j + D^{i+1\,j+1}_{i+2\,j+2} y_j) \end{aligned} \right\} \quad (i = 0, 1, 2).$$

If we make Σ_1 correspond to S, we may verify at once that $\Sigma_1' \Sigma_1$ will correspond to $S'S$. We use our earlier formula in the form

$$D''^{i\,j}_{k\,l} = \sum_s^{0,1,2} (D^{i\,s}_{k\,3} D'^{s\,j}_{3\,l} + D^{i\,s}_{k\,s+1} D'^{s\,j}_{s+1\,l}),$$

where $s+1$ is taken modulo 3.

15. Theorem. — *The substitution* Σ_1 *merely multiplies the function* $\sum_i^{0,1,2} x_i y_i$ *by the determinant* Δ_6 *of* Σ_1.

The conditions are as follows (see § 1) :—

$$(1) \quad \sum_j^{0,1,2} D^{i\,j}_{3\,3} D^{i\,j+1}_{3\,j+2} = \sum_j^{0,1,2} D^{i+1\,j}_{i+2\,3} D^{i+1\,j+1}_{i+2\,j+2} = 0.$$

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* This is equivalent to transforming Σ by a substitution of determinant $a \equiv \pm 1$.

These are algebraic identities, being one-half of the expansions of the determinants

$$\left|\begin{array}{cccc} a_{i0} & a_{i1} & a_{i2} & a_{i3} \\ a_{30} & a_{31} & a_{32} & a_{33} \\ \hline a_{i0} & a_{i1} & a_{i2} & a_{i3} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{array}\right|, \quad \left|\begin{array}{cccc} a_{i+1\,0} & a_{i+1\,1} & a_{i+1\,2} & a_{i+1\,3} \\ a_{i+2\,0} & a_{i+2\,1} & a_{i+2\,2} & a_{i+2\,3} \\ \hline a_{i+1\,0} & a_{i+1\,1} & a_{i+1\,2} & a_{i+1\,3} \\ a_{i+2\,0} & a_{i+2\,1} & a_{i+2\,2} & a_{i+2\,3} \end{array}\right|.$$

$$(2) \qquad \sum_j^{0,1,2} \left(D_{33}^{ij} D_{k+2\,j+2}^{k+1\,j+1} + D_{3\,j+2}^{i\,j+1} D_{k+2\,3}^{k+1\,j}\right) = \delta_{ik}\Delta_6 \quad (i, k = 0, 1, 2),$$

where $\delta_{ii} = 1$, $\delta_{ik} = 0$ for $i \neq k$.

The left side of (2) is the expansion of

$$\left|\begin{array}{cccc} a_{i0} & a_{i1} & a_{i2} & a_{i3} \\ a_{30} & a_{31} & a_{32} & a_{33} \\ \hline a_{k+1\,0} & a_{k+1\,1} & a_{k+1\,2} & a_{k+1\,3} \\ a_{k+2\,0} & a_{k+2\,1} & a_{k+2\,2} & a_{k+2\,3} \end{array}\right| = +\delta_{ik}\Delta_6$$

for every $i, k = 0, 1, 2$.

16. THEOREM.—*The substitution Σ_1 can reduce to the form*

$$T_{0,\nu}: \begin{Bmatrix} x_0' = \nu x_0, & x_i' = x_i \\ y_0' = \nu^{-1} y_0, & y_i' = y_i \end{Bmatrix} \quad (i = 1, 2),$$

if, and only if, ν be a square in the $GF[p^n]$.

Consider the "partial" transformation (whose determinant may be zero) derived from Σ_1,

$$\Sigma_y: \; y_i' = \sum_j^{0,1,2} D_{i+2\,j+2}^{i+1\,j+1} y_j \quad (i = 0, 1, 2).$$

Its determinant is seen to equal

$$\Delta_3^2 \equiv \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix}^2.$$

The analogous substitution derived from $T_{0,\nu}$ is

$$\begin{Bmatrix} \nu^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix}$$

of determinant ν^{-1}, which must therefore be a square.

Inversely, if ν be a square, to the substitution

$$S = \begin{Bmatrix} \nu^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & \nu^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \nu^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & \nu^{\frac{1}{2}} \end{Bmatrix}$$

corresponds the substitution $\Sigma_1 = T_{0,\nu}$.

17. Theorem.—*To the substitution* $\Sigma_1 = 1$ *correspond exactly the two substitutions* $S = \pm 1$, *multiplying every index by the constant* ± 1.

Indeed the "partial" transformation

$$(\Delta_3^{-1} D^{i+1\,j+1}_{i+2\,j+2}) \quad (i, j = 0, 1, 2)$$

is the reciprocal of the transformation of determinant Δ_3,

$$\begin{Bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{Bmatrix};$$

so that, for $\Sigma_1 = 1$, the latter must have the form

$$\begin{Bmatrix} \Delta_3 & 0 & 0 \\ 0 & \Delta_3 & 0 \\ 0 & 0 & \Delta_3 \end{Bmatrix}.$$

By comparing their determinants, we have

$$\Delta_3 = \Delta_3^3, \quad \Delta_3 = \pm 1.$$

Further, the determinant of Σ_1 being the cube of the determinant of S, the latter must be $+1$. Hence

$$a_{33} = \pm 1.$$

Building the second minors of

$$(a_{ij}) \quad (i, j = 0, 1, 2, 3),$$

which have one and but one element from the main diagonal, we find that

$$a_{3j} = a_{i3} = 0 \quad (i, j = 0, 1, 2).$$

Corollary.—If we consider the substitution (a_{ij}) to be identical with $(-a_{ij})$, and denote the pair by S_1, we obtain a simple isomorphism between the groups $\{S_1\}$ and $\{\Sigma_1\}$.

The group of the S_1 is of index 1 or 2 under the group of all quaternary linear substitutions in the $GF[p^n]$ according as $p = 2$ or $p > 2$.

18. Denote by $G_{4,n,p}$ the group of quaternary linear substitutions S in the $GF[p^n]$ which have the determinant unity. Its order* is

$$(p^{4n}-1)(p^{4n}-p^n)(p^{4n}-p^{2n})\,p^{3n}.$$

Denote the corresponding substitution Σ_1 of determinant unity by Σ_1'. Hence, for $p > 2$, the order of the group $\{\Sigma_1'\}$ is

$$\tfrac{1}{2}\,(p^{3n}-1)(p^{4n}-1)\,p^{4n}\,(p^{2n}-1)\,p^{2n}.$$

This being the order of the group $L''_{3,n,p}$, which $T_{i,\nu}$ ($\nu =$ not-square in the $GF[p^n]$) extends to the group $L'_{3,n,p}$ of all senary linear substitutions of determinant unity in the $GF[p^n]$, leaving $\Sigma x_i y_i$ invariant, it follows, from the theorem of § 16, that $\{\Sigma_1'\} \equiv L''_{3,n,p}$.

Likewise, for $p = 2$, the order of the group $\{\Sigma_1'\}$ is

$$(2^{3n}-1)(2^{4n}-1)\,2^{4n}\,(2^{2n}-1)\,2^{2n}.$$

This equals the order of the group $L'_{3,n,2} \equiv L''_{3,n,2}$ of all senary linear substitutions in the $GF[2^n]$, which leave invariant $\Sigma x_i y_i$, and which, if written in the form (1) of § 1, satisfy the relation

$$\sum_{i,j}^{1,2,3} a_{ij}\delta_{ij} = 1.$$

Writing the left member in the notation used for our substitutions Σ_1, it becomes

$$\sum_{i,j}^{0,1,2} D_{3\,3}^{i\,j}\, D_{i+2\,j+2}^{i+1\,j+1}$$

$$\equiv D_{33}^{00}D_{22}^{11} + D_{33}^{01}D_{20}^{12} + D_{33}^{02}D_{21}^{10} + D_{33}^{10}D_{02}^{21} + D_{33}^{11}D_{00}^{22} + D_{33}^{12}D_{01}^{20}$$
$$+ D_{33}^{20}D_{12}^{01} + D_{33}^{21}D_{10}^{02} + D_{33}^{22}D_{11}^{00},$$

* *Annals of Mathematics*, October, 1897.

which may be given the form

$$\begin{vmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{vmatrix} + 2a_{33} \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix}.$$

Hence, for every p, the group $\{\Sigma_1'\}$ coincides with the group $L''_{3,n,p}$.

19. From the known* structure of $G_{4,n,p}$, we readily deduce that of the isomorphic group $\{\Sigma_1'\}$. The quaternary group of substitutions of determinant unity has the maximal invariant sub-group generated by the substitution M,

$$M = \begin{Bmatrix} \mu & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{Bmatrix}, \quad \mu^4 = 1, \quad \mu^{p^n-1} = 1.$$

To M corresponds in $\{\Sigma_1'\}$ the substitution which multiplies every index by μ^2. Hence,

(1) If $p = 2$, μ has the value 1 only, so that $\{\Sigma_1'\} \equiv L''_{3,n,2}$ is simple.

(2) If $p^n = 4t+3$, so that -1 is a not-square in the $GF[p^n]$, $\mu = \pm 1$. Hence $L''_{3,n,p}$ is simple.

(3) If $p^n = 4t+1$, so that $-1 =$ square, $\mu = \pm 1$ or $\pm\sqrt{-1}$. Hence $L''_{3,n,p}$ has the maximal invariant sub-group of order 2, generated by the substitution multiplying every index by -1.

The above results agree with those obtained earlier in the paper for the general group $L''_{m,n,p}$.

20. Furthermore, our investigation gives an immediate proof of the following theorem for continuous groups :—

The quaternary group of all real linear homogeneous transformations of determinant unity is hemihedrically isomorphic with the senary group of all real linear homogeneous transformations of determinant unity which leave invariant a homogeneous quadratic function.

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* *Annals of Mathematics*, October, 1897.

The corresponding projective groups are simply isomorphic, and each is a simple group of fifteen parameters—results due to Lie. Moreover, to every projective transformation (finite or infinitesimal) of ordinary space, we obtain at once the corresponding projective transformation leaving invariant the surface

$$\sum_{i=0}^{2} x_i y_i = 0,$$

and hence that leaving invariant an arbitrary non-degenerate surface of the second order in space of five dimensions.

To obtain the general infinitesimal transformation of the form Σ_1, we set

$$a_{ii} = 1 + a_{ii}\delta t, \quad a_{ij} = a_{ij}\delta t \quad (i, j = 0, \ldots, 3).$$

We readily obtain the following expressions:—*

$$\delta x_i \equiv x_i' - x_i = \delta t \left\{(a_{33} + a_{ii}) x_i + a_{i\,i-1} x_{i-1} + a_{i\,i+1} x_{i+1} + a_{3,\,i+1} y_{i-1} - a_{3,\,i+2} y_{i+1}\right\},$$

$$\delta y_i \equiv y_i' - y_i = \delta t \left\{(a_{i+1\,i+1} + a_{i+2\,i+2}) y_i - a_{i+1\,i} y_{i+1} - a_{i+2\,i} y_{i+2} + a_{i+2\,3} x_{i+1} - a_{i+1\,3} x_{i+2}\right\}.$$

$$\delta\left(\sum_{i=0}^{2} x_i y_i\right) = (a_{00} + a_{11} + a_{22} + a_{33}) \sum_{i=0}^{2} x_i y_i.$$

The condition for absolute invariance of $\Sigma x_i y_i$ is thus the same as the condition derived from

$$|\, a_{ij} \,| = 1.$$

It is seen that the above infinitesimal transformation contains all sixteen arbitrary parameters a_{ij}.

21. THEOREM.—*For $p^n = 4l+1$, the group $L_{m,n,p}$ is simply isomorphic with the group*† *of orthogonal substitutions on $2m$ indices in the $GF[p^n]$.*

Since -1 is in this case the square of a quantity I of the $GF[p^n]$, we may introduce the new indices X_i, Y_i defined thus:

$$\xi_i = X_i + IY_i, \quad \eta_i = X_i - IY_i \quad (i = 1, \ldots, m).$$

Then
$$\sum_{i=1}^{m} \xi_i \eta_i = \sum_{i=1}^{m} (X_i^2 + Y_i^2).$$

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* As usual, the subscripts i, $i \pm 1$, $i \pm 2$ are to be taken modulo 3.

† The writer has determined the structure of the orthogonal group on m indices in the $GF[p^n]$ for every integer $m \neq 4$, n, and prime number p: *Proceedings of the California Academy of Sciences*, Third Series, Vol. I., Nos. 4 and 5; *Bulletin of the Amer. Math. Soc.*, February and May, 1898.

22. Consider the group of linear substitutions (1) on m pairs of indices in the $GF[p^n]$ which have the absolute invariant

$$\phi \equiv \sum_{i=1}^{m} \xi_i \eta_i + \sum_{i=1}^{m} (a_i \xi_i + \gamma_i \eta_i),$$

where* the a_i and γ_i are not all zero.

If a substitution T converts ϕ into Φ, the group leaving ϕ invariant is the transformed by T of the group leaving Φ invariant. Taking as T the following product belonging to the group $L_{m,n,p}$ leaving $\sum_{i=1}^{m} \xi_i \eta_i$ invariant:

$$T \equiv W_{1,\,m,\,-\gamma_m} Q_{1,\,m,\,-a_m} \dots W_{1,\,2,\,-\gamma_2} Q_{1,\,2,\,-a_2} T_{1,\,a_1^{-1}},$$

we find that
$$\Phi \equiv T(\phi) = \sum_{i=1}^{m} \xi_i \eta_i + \xi_1 + \lambda \eta_1,$$

where
$$\lambda \equiv \sum_{i=1}^{m} a_i \gamma_i.$$

23. Suppose first that $p^n = 2$. Then $\lambda = 0$ or 1.

(*a*) For $\lambda = 1$, the group leaving Φ invariant is the second hypo-Abelian group G_1 (Jordan, *Traité des Substitutions*, pp. 206–213 and 440). For $m > 1$, G_1 has a simple sub-group of index 2 and of order

$$(2^m+1)(2^{2m-2}-1)\,2^{2m-2}\,(2^{2m-4}-1)\,2^{2m-4} \dots (2^2-1)\,2^2,$$

which is composed of all the substitutions of G_1 which satisfy the relation†

$$\sum_{i,j}^{1,\dots,m} a_{ij}\delta_{ij} + a_{11} + \beta_{11} + \gamma_{11} + \delta_{11} \equiv m \pmod 2.$$

(*b*) For $\lambda = 0$, we introduce the new indices

$$\left.\begin{array}{ll} \bar{\eta}_1 = \eta_1 + \xi_1, & \bar{\eta}_i = \eta_i \\ \bar{\xi}_1 = \xi_1, & \bar{\xi}_i = \xi_i \end{array}\right\} \quad (i = 2, \dots, m).$$

Then
$$\sum_{i=1}^{m} \bar{\xi}_i \bar{\eta}_i \equiv \sum_{i=1}^{m} \xi_i \eta_i + \xi_1.$$

This transformation of indices is equivalent to the Abelian substitution

* We will suppose that $a_1 \neq 0$, changing, if necessary, the notation, which is allowable on account of the symmetry of $\Sigma \xi_i \eta_i$.

† "The Structure of the Hypo-Abelian Groups," *Bulletin of the Amer. Math. Soc.*, July, 1898.

L_1' (Jordan, p. 174). Hence the group with the invariant $\Phi_{\lambda=0}$ is conjugate within the Abelian group $H_{m,1,2}$ in the $GF[2]$ to the first hypo-Abelian group $L_{m,1,2}$ defined in § 1 (the group G_0 of Jordan, p. 199).

Combining the results of cases (a) and (b), we have proved the following

THEOREM.—*The group of linear substitutions modulo 2 on $2m$ indices which leave invariant*

$$\sum_{i=1}^{m} (\xi_i\eta_i + a_i\xi_i + \gamma_i\eta_i)$$

is conjugate within the Abelian group to the first or else to the second hypo-Abelian group.

24. For $p^n > 2$ the group with the invariant Φ must leave separately invariant

$$\sum_{i=1}^{m} \xi_i\eta_i, \quad \xi_1 + \lambda\eta_1 ;$$

and is therefore a sub-group of the group $L_{m,n,p}$. We will designate the group as $B_{m,n,p}$. For $m > 1$, it contains (§ 2) a sub-group $B'_{m,n,p}$ of index 2, which the transposition $(\xi_m\eta_m)$ extends to $B_{m,n,p}$. We treat in turn the cases $\lambda = 0$ and $\lambda \neq 0$.

25. For $\lambda = 0$ the substitutions of $B'_{m,n,p}$ have the form

$$S: \left\{\begin{aligned} &\xi_1' = \xi_1, \quad \eta_1' = \beta_{11}\xi_1 + \eta_1 + \sum_{j=2}^{m} (\beta_{1j}\xi_j + \delta_{1j}\eta_j) \\ &\xi_i' = a_{i1}\xi_1 + \sum_{j=2}^{m} (a_{ij}\xi_j + \gamma_{ij}\eta_j) \\ &\eta_i' = \beta_{i1}\xi_1 + \sum_{j=2}^{m} (\beta_{ij}\xi_j + \delta_{ij}\eta_j) \end{aligned}\right\}$$

$$(i = 2, \ldots, m),$$

where

$$\beta_{11} + \sum_{j=2}^{m} \beta_{1j}\delta_{1j} = 0, \tag{β}$$

and where the $a_{ij}, \gamma_{ij}, \beta_{ij}, \delta_{ij}$ $(i, j = 2, \ldots, m)$ are subject to the same conditions as the substitution*

$$S_1: \left\{\begin{aligned} &\xi_i' = \sum_{j=2}^{m} (a_{ij}\xi_j + \gamma_{ij}\eta_j) \\ &\eta_1' = \sum_{j=2}^{m} (\beta_{ij}\xi_j + \delta_{ij}\eta_j) \end{aligned}\right\} \quad (i = 2, \ldots, m)$$

---

* The determinant of S_1 equals the determinant of S.

belonging to the group $L'_{m-1,n,p}$. To prove this statement and its inverse, we note that $B'_{m,n,p}$ contains every product of the form (see § 2)

$$R \equiv V_{2,1,-\beta_{12}} Q_{2,1,-\delta_{12}} \dots V_{m,1,-\beta_{1m}} Q_{m,1,-\delta_{1m}}$$

which leaves ξ_1 fixed and replaces η_1 by

$$\beta_{11}\xi_1 + \eta_1 + \sum_{j=2}^{m} (\beta_{1j}\xi_j + \delta_{1j}\eta_j),$$

provided (β) holds. Hence every substitution of $B'_{m,n,p}$ is of the form RS_1. The number of solutions in the $GF[p^n]$ of (β) being $p^{n(2m-2)}$, the order of $B'_{m,n,p}$ equals $p^{n(2m-2)} O'_{m-1,n,p}$, the latter factor being the order of $L'_{m-1,n,p}$.

To each substitution S there corresponds a single one S_1. We readily verify that to the product SS' there corresponds the product $S_1 S'_1$ of the corresponding substitutions. Hence,

THEOREM.—*The group $B'_{m,n,p}$ is isomorphic with the group $L'_{m-1,n,p}$; to the substitution* 1 *of the latter correspond the $p^{n(2m-2)}$ substitutions R of the former.*

Hence the group of the substitutions R which is generated by the $2(m-1)$ *commutative* substitutions

$$V_{i,1,\lambda}, \quad Q_{i,1,\lambda} \quad \begin{pmatrix} i = 2, \dots, m \\ \lambda \text{ arbitrary in } GF[p^n] \end{pmatrix}$$

is an invariant sub-group of $B'_{m,n,p}$ of index $p^{n(2m-2)}$, the quotient group being $L'_{m-1,n,p}$.

This last result follows at once from the corollary to § 2. Thus the $Q_{i,j,\lambda}$ and S_{ij} ($i, j, = 2, \dots, m$), together with some one $T_{2,\nu}$ (ν = not-square), generate the group $L'_{m-1,n,p}$. We have therefore only to verify the formulæ

$$Q^{-1}_{i,j,a} Q_{i,1,\lambda} Q_{i,j,a} = Q_{i,1,\lambda},$$

$$Q^{-1}_{j,i,a} Q_{i,1,\lambda} Q_{j,i,a} = Q_{i,1,\lambda} Q_{j,1,a\lambda},$$

$$Q^{-1}_{i,j,a} V_{i,1,\lambda} Q_{i,j,a} = V_{i,1,\lambda} V_{j,1,-a\lambda};$$

$$Q^{-1}_{j,i,a} V_{i,1,\lambda} Q_{j,i,a} = V_{i,1,\lambda}.$$

$$S^{-1}_{ij} Q_{i,1,\lambda} S_{ij} = V_{i,1,\lambda},$$

$$T^{-1}_{2,\nu} Q_{2,1,\lambda} T_{2,\nu} = Q_{2,1,\lambda\nu}.$$

26. Consider the group $B'_{m,n,p}$ with the invariants

$$\sum_{i=1}^{m} \xi_i \eta_i, \quad \xi_1+\lambda\eta_1 \quad (\lambda \neq 0).$$

Its substitutions (1) must satisfy the relations (2), together with the following :—

$$(3) \quad \begin{cases} a_{11}+\lambda\beta_{11} = 1, & a_{1j}+\lambda\beta_{1j} = 0 \\ \gamma_{11}+\lambda\delta_{11} = \lambda, & \gamma_{1j}+\lambda\delta_{1j} = 0 \end{cases} \quad (j = 2, \ldots, m).$$

Writing the same relations for the inverse of (1),

$$(3)_{-1} \quad \begin{cases} \delta_{11}+\lambda\beta_{11} = 1, & \delta_{j1}+\lambda\beta_{j1} = 0 \\ \gamma_{11}+\lambda a_{11} = \lambda, & \gamma_{j1}+\lambda a_{ji} = 0 \end{cases} \quad (j = 2, \ldots, m).$$

27. THEOREM.—*The group $B'_{m,n,p}$ results from the extension of $L'_{m-1,n,p}$ on the indices ξ_i, η_i $(i = 2, \ldots, m)$ by the substitutions (which leave $\xi_1+\lambda\eta_1$ invariant)*

$$S_{12}T_{1\lambda}T_{2\lambda} \quad \textit{and} \quad Q_{2,1,a}W_{1,2,a\lambda}.$$

Let an arbitrary substitution S of $B'_{m,n,p}$ replace ξ_1 by

$$f_1 \equiv \sum_{j=1}^{m} (a_{1j}\xi_j+\gamma_{1j}\eta_j),$$

where $\quad$ (4) $\quad \gamma_{11}+\lambda a_{11} = \lambda, \quad \sum_{j=1}^{m} a_{1j}\gamma_{1j} = 0.$

From the substitutions $S_{12}T_{1\lambda}T_{2\lambda}$, $Q_{2,1,a}\,.\,W_{1,2,a\lambda}$, and those of the group $L'_{m-1,n,p}$, all of which belong to $B'_{m,n,p}$, we can derive a substitution T which shall replace ξ_1 by f_1.

(*a*) If $a_{1i} = \gamma_{1i} = 0$ $(i = 2, \ldots, m)$, we have $f_1 = \xi_1$ or $\lambda\eta_1$, so that we may take $T = 1$ or $S_{12}T_{1\lambda}T_{2\lambda}$ respectively.

(*b*) If $a_{12} \neq 0$, for example, the group $L'_{m-1,n,p}$ contains a substitution T' replacing η_2 by

$$a_{12}\xi_2+\left(\gamma_{12}+\frac{a_{11}\gamma_{11}}{a_{12}}\right)\eta_2+\sum_{j=3}^{m}(a_{1j}\xi_j+\gamma_{1j}\eta_j)\,;$$

indeed, by (4), the necessary condition is satisfied,

$$a_{12}\left(\gamma_{12}+\frac{a_{11}\gamma_{11}}{a_{12}}\right)+\sum_{j=3}^{m} a_{1j}\gamma_{1j} = 0.$$

We may thus take for T the product

$$Q_{2,1,-\lambda^{-1}\gamma_{11}a_{12}^{-1}}\,W_{1,2,-\gamma_{11}a_{12}^{-1}}\,T'Q_{2,1,\lambda^{-1}}\,W_{1,2,1},$$

which replaces ξ_1 by

$$f_1 \equiv (1-\lambda^{-1}\gamma_{11})\,\xi_1+\gamma_{11}\eta_1+a_{12}\xi_2+\left[\gamma_{12}+\gamma_{11}a_{12}^{-1}(a_{11}+\lambda^{-1}\gamma_{11}-1)\right]\eta_2 + \sum_{j=3}^{m}(a_{1j}\xi_j+\gamma_{1j}\eta_{1j}).$$

Thus, for both cases (a) and (b), we may set $S = TT_1$, where T_1 leaves fixed ξ_1, and therefore also η_1, so that T_1 belongs to $L'_{m-1,n,p}$.

28. Theorem.—*The order of $B'_{m,n,p}$ is*

$$(p^{n(2m-2)}-1)\,p^{n(2m-3)}\,(p^{n(2m-4)}-1)\,p^{n(2m-5)}\dots(p^{2n}-1)\,p^n.$$

Indeed, the order is the product of $O'_{m-1,n,p}$ by $R_{m,n,p}$, the number of solutions in the $GF[p^n]$ of the pair of equations (4), λ being a fixed quantity $\neq 0$. Eliminating γ_{11} from (4), we have

$$\lambda a_{11}(1-a_{11})+a_{12}\gamma_{12}+\dots+a_{1m}\gamma_{1m}=0.$$

For $a_{1m}\gamma_{1m}=0$ this equation has

$$(2\,.\,p^n-1)\,R_{m-1,n,p}$$

solutions. For $a_{1m}\gamma_{1m}\neq 0$ the expression

$$\lambda a_{11}(1-a_{11})+a_{12}\gamma_{12}+\dots+a_{1\,m-1}\gamma_{1\,m-1}$$

has $(p^{n(2m-3)}-R_{m-1,n,p})$ values $\mu\neq 0$, for each of which $a_{1m}\gamma_{1m}+\mu=0$ has (p^n-1) sets of solutions. Hence we have the recursion formula

$$R_{m,n,p}=(p^n-1)\,p^{n(2m-3)}+p^n R_{m-1,n,p},$$

with the initial value $R_{1,n,p}=2$. By induction,

$$R_{m,n,p}=p^{n(m-1)}\,(p^{n(m-1)}+1).$$

29. Theorem.—*Within the group $L'_{m,n,p}$, the groups $B'_{m,n,p}$ for which λ is a square are all conjugate; likewise the groups $B'_{m,n,p}$ for which λ is a not-square are conjugate.*

Indeed, on applying to $\xi_1+\lambda\eta_1$ the substitution $T_{1,\mu}$ belonging to $L'_{m,n,p}$, we obtain

$$\mu\,(\xi_1+\lambda\mu^{-2}\eta_1).$$

30. Theorem.—*Within the group $L'_{m,2n,p}$ in the $GF[p^{2n}]$ all the groups $B'_{m,n,p}$ are conjugate.*

The substitution $T\equiv T_{1,\lambda^{\frac12}}\,T_{2,\lambda^{\frac12}}\dots T_{m,\lambda^{\frac12}}$

belongs to the $GF[p^{2n}]$ if λ be a not-square. T leaves $\sum_{i=1}^{m}\xi_i\eta_i$ fixed,

and converts $\xi_1+\lambda\eta_1$ into $\lambda^{\frac{1}{2}}(\xi_1+\eta_1)$. Further, T transforms the general substitution of the $GF[p^n]$

$$S: \left\{\begin{array}{l} \xi_i' = \sum_{j=1}^{m} (a_{ij}\xi_j+\gamma_{ij}\eta_j) \\ \eta_i' = \sum_{j=1}^{m} (\beta_{ij}\xi_j+\delta_{ij}\eta_j) \end{array}\right\} \qquad (i=1, \ldots, m)$$

which leaves $\Sigma\xi_i\eta_i$ and $\xi_1+\eta_1$ invariant into

$$T^{-1}ST: \left\{\begin{array}{l} \xi_i' = \sum_{j=1}^{m} (a_{ij}\xi_j+\lambda\gamma_{ij}\eta_j) \\ \eta_i' = \sum_{j=1}^{m} (\lambda^{-1}\beta_{ij}\xi_j+\delta_{ij}\eta_j) \end{array}\right\} \qquad (i=1, \ldots, m),$$

which belongs to the $GF[p^n]$, and leaves invariant

$$\sum_{i=1}^{m} \xi_i\eta_i, \quad \xi_1+\lambda\eta_1.$$

We denote by $M'_{m,n,p}$ that particular group $B'_{m,n,p}$ for which $\lambda = 1$. Its order equals the order of the Abelian* group $H_{m-1,n,p}$ on $m-1$ pairs of indices in the $GF[p^n]$. For $p=2$, $H_{m,n,2}$ contains $M'_{m,n,2}$ as a sub-group.

31. Theorem.—*The groups* $M'_{m,n,2}$ *and* $H_{m-1,n,2}$ *are simply isomorphic.*

If we take† $\xi \equiv \xi_1+\eta_1$ as a new index in place of ξ_1 and apply the relations (3) and $(3)_{-1}$ for $\lambda=1$, the general substitution of $M'_{m,n,}$ takes the form S of § 25. The theorem follows by making the abridged substitution S_1 correspond to S.

The fact that the groups have the same order also follows from the isomorphism. Indeed, from the relations (3), $(3)_{-1}$, and the first one of the set (2), it readily follows that, if S_1 be the identity, so is also S.

32. Theorem. — *For* $p^n = 4l+1$, *the group* $M'_{m,n,p}$ *is simply isomorphic with the group of orthogonal substitutions of determinant unity on* $2m-1$ *indices in the* $GF[p^n]$.

In the case considered, -1 is the square of a mark I of the $GF[p^n]$. Introduce new indices X, Y, X_i, Y_i defined as follows:—

$$\xi_1 = X+Y, \qquad \eta_1 = -X+Y,$$
$$\xi_i = IX_i+Y_i, \quad \eta_i = IX_i-Y_i \quad (i=2, \ldots, m).$$

* Dickson, "A Triply Infinite System of Simple Groups," *The Quarterly Journal of Pure and Applied Mathematics*, July, 1897.

† This transformation of indices corresponds to the Abelian substitution L_1. Hence the transformed substitutions are also Abelian.

The group $M'_{m,n,p}$ has the invariants

$$\xi_1+\eta_1 \equiv 2Y,$$
$$\sum_{i=1}^{m} \xi_i\eta_i \equiv Y^2-(X^2+X_2^2+Y_2^2+\ldots+X_m^2+Y_m^2).$$

Expressed in terms of the new indices, every substitution of $M'_{m,n,p}$ is seen to be independent of the index Y, and therefore a linear homogeneous substitution on the indices X, Y_i, X_i leaving invariant

$$X^2+\sum_{i=2}^{m}(X_i^2+Y_i^2).$$

33. THEOREM.*—*The quaternary group $M'_{2,n,p}$ has a sub-group of index 2, which is simply isomorphic with the group of linear fractional substitutions of determinant unity in the $GF[p^n]$.*

Applying (3) and $(3)_{-1}$ to certain of the relations (2), we get

$$\alpha_{12}^2=\beta_{12}^2=\alpha_{22}\beta_{22},\quad \beta_{21}^2=\delta_{21}^2=\beta_{22}\delta_{22},$$
$$\gamma_{12}^2=\delta_{12}^2=\gamma_{22}\delta_{22},\quad \alpha_{21}^2=\gamma_{21}^2=\alpha_{22}\gamma_{22}.$$

It follows that α_{22}, β_{22}, γ_{22}, δ_{22} are all squares or all not-squares in the $GF[p^n]$. This is evident if not more than one of them be zero; while for two of them zero it follows, since

$$\alpha_{22}\delta_{22}+\beta_{22}\gamma_{22}=1.$$

We may therefore take them to be $\nu\alpha^2$, $\nu\beta^2$, $\nu\gamma^2$, $\nu\delta^2$, respectively, where ν is either unity or some particular not-square. We may then set

$$\alpha_{21}=\nu\alpha\gamma,\quad \beta_{21}=\nu\beta\delta,$$

taking the ambiguities of sign into α and δ, for example. In virtue of the relations

$$\alpha_{11}\beta_{11}+\alpha_{21}\beta_{21}=0,\quad \alpha_{11}\gamma_{11}+\alpha_{12}\gamma_{12}=0,\quad \beta_{11}=\gamma_{11},$$

the signs in $$\alpha_{12}=\pm\nu\alpha\beta,\quad \gamma_{12}=\pm\nu\gamma\delta$$

[* It was suggested by the Referees that this theorem should follow readily from the known theorems on the real projective transformations of a real ruled quadric. Indeed, those projective transformations which leave the generators of one set individually unchanged form a group simply isomorphic with the linear fractional group in the variable parameter giving the generators of the other set. It seems to the writer, however, that the carrying over of theorems on projective groups into theorems on groups in a Galois field must be employed with considerable caution; the reverse process is certainly impossible in general. The process is probably limited to the case in which the theorem holds in an arbitrary Galois field.—*January* 28*th*, 1899.]

must agree. Writing a for $\pm a$, γ for $\pm\gamma$, the expressions for a_{21} and β_{21} remain unaltered, and the general substitution of $M'_{2,n,p}$ takes the form

$$\left\{\begin{array}{cccc} a_{11} & 1-a_{11} & \nu a\beta & \nu\gamma\delta \\ 1-a_{11} & a_{11} & -\nu a\beta & -\nu\gamma\delta \\ \nu a\gamma & -\nu a\gamma & \nu a^2 & \nu\gamma^2 \\ \nu\beta\delta & -\nu\beta\delta & \nu\beta^2 & \nu\delta^2 \end{array}\right\}.$$

The relations (2) now reduce to the following:—

$$\nu^2(a\delta-\beta\gamma)^2 = 1, \quad a_{11}(a_{11}-1) = \nu^2 a\beta\gamma\delta,$$
$$\rho\left\{1-2a_{11}+\nu(a\delta+\beta\gamma)\right\} = 0,$$

where ρ may be successively $\nu a\gamma$, $\nu a\beta$, $\nu\beta\delta$, $\nu\gamma\delta$. If these do not vanish simultaneously, the three relations combine to give either

$$(1)\ \left\{\begin{array}{l} a_{11} = \nu a\delta, \\ +1 = \nu(a\delta-\beta\gamma), \end{array}\right.$$

or

$$(2)\ \left\{\begin{array}{l} a_{11} = \nu\beta\gamma, \\ -1 = \nu(a\delta-\beta\gamma). \end{array}\right.$$

The determinant of the above substitution is

$$D \equiv \nu^2(a^2\delta^2-\beta^2\gamma^2)-2\left|\begin{array}{ccc} 1-a_{11} & -\nu a\beta & -\nu\gamma\delta \\ \nu a\gamma & \nu a^2 & \nu\gamma^2 \\ \nu\beta\delta & \nu\beta^2 & \nu\delta^2 \end{array}\right|.$$

Applying (1) and (2), this becomes respectively

$$D = \nu^2(a^2\delta^2-\beta^2\gamma^2)-2\nu^3\beta\gamma(a\delta-\beta\gamma)^2 = \nu(a\delta-\beta\gamma) = 1,$$
$$D = \nu^2(a^2\delta^2-\beta^2\gamma^2)+2\nu^3 a\delta(a\delta-\beta\gamma)^2 = \nu(a\delta-\beta\gamma) = -1.$$

Thus, if $p>2$, the alternative (2) must be excluded, since the substitutions of M' have the determinant $+1$. For $p=2$, the relation

$$\sum_{i,j}^{1\ldots m} a_{ij}\delta_{ij} = m$$

reduces to $$a_{11} = \nu a\delta.$$

Next, we suppose that all four values of ρ vanish. Then $a_{11}=0$ or 1, while, by inspection,

$$D = (2a_{11}-1)(\nu^2a^2\delta^2-\nu^2\beta^2\gamma^2).$$

Since $\quad D = +1 \quad$ and $\quad \nu^2 (\alpha\delta - \beta\gamma)^2 = 1,$

we have, for $a_{11} = 1, \qquad \nu^2\alpha^2\delta^2 = 1, \quad \nu\beta\gamma = 0;$

for $a_{11} = 0, \qquad \nu^2\beta^2\gamma^2 = 1, \quad \nu\alpha\delta = 0.$

We may take the ambiguities of sign into α and γ, for example, so that $\nu\alpha\delta = +1$, $\nu\beta\gamma = 0$, when $a_{11} = 1$; while $\nu\alpha\delta = 0$, $\nu\beta\gamma = -1$, when $a_{11} = 0$. Hence we are always led to the alternative (1), so that every substitution of M' may be given the form

$$\begin{Bmatrix} \nu\alpha\delta & -\nu\beta\gamma & \nu\alpha\beta & \nu\gamma\delta \\ -\nu\beta\gamma & \nu\alpha\delta & -\nu\alpha\beta & -\nu\gamma\delta \\ \nu\alpha\gamma & -\nu\alpha\gamma & \nu\alpha^2 & \nu\gamma^2 \\ \nu\beta\delta & -\nu\beta\delta & \nu\beta^2 & \nu\delta^2 \end{Bmatrix},$$

where $$\nu\,(\alpha\delta - \beta\gamma) = 1.$$

Giving it the notation $\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}_\nu$, we may verify that two such substitutions compound as follows :—

$$\begin{bmatrix} \alpha' & \gamma' \\ \beta' & \delta' \end{bmatrix}_{\nu'} \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}_{\nu} = \begin{bmatrix} \alpha'\alpha+\beta'\gamma & \gamma'\alpha+\delta'\gamma \\ \alpha'\beta+\beta'\delta & \gamma'\beta+\delta'\delta \end{bmatrix}_{\nu'\nu}.$$

Hence the sub-group of $M'_{2,n,p}$, of index 2, formed of those substitutions in which $\nu = 1$ (or a square) is simply isomorphic with the group of linear fractional substitutions in the $GF[p^n]$ having the determinant $+1$.

34. Consider the largest sub-group $K_{m,n,p}$ common to the Abelian group $H_{m,n,p}$ and the group $L_{m,n,p}$. The inverse* of a substitution (1) of H is obtained by replacing $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij}$ by respectively δ_{ji}, $-\beta_{ji}$, $-\gamma_{ji}$, α_{ji}. The inverse of a substitution (1) of L is obtained by replacing the same by δ_{ji}, $+\beta_{ji}$, $+\gamma_{ji}$, α_{ji}, respectively. Hence, if $p > 2$, every substitution (1) of K has

$$\beta_{ji} = \gamma_{ji} = 0 \quad (i, j = 1, \ldots, m);$$

and therefore is the product of the two dualistic substitutions

$$\xi'_i = \sum_{j=1}^{m} \alpha_{ij}\xi_j, \quad \eta'_i = \sum_{j=1}^{m} \delta_{ij}\eta_j \quad (i = 1, \ldots, m),$$

* Jordan, § 218.

where δ_{ij} is the adjoint of a_{ij} in the determinant $| a_{ij} |$. K is thus generated by the $T_{i,\lambda}$ and $Q_{i,j,\lambda}$. Its order and structure follow from that of the general linear homogeneous group* on m indices in the $GF[p^n]$.

35. THEOREM.—*For $p > 2$ the largest sub-group common to $H_{m,n,p}$ and $B_{m,n,p}$ (for $\lambda = 1$) is $K_{m-1,n,p}$.*

It is clearly that sub-group of $K_{m,n,p}$ which has the invariant $\xi_1 + \eta_1$. Hence

$$a_{11} = \delta_{11} = 1, \quad a_{1j} = \delta_{1j} = 0 \quad (j = 2, \ldots, m).$$

Writing these relations for the inverse of (1), we have

$$\delta \ = a_{j1} = 0 \quad (j = 2, \ldots, m).$$

The substitutions of the sub-group have therefore the form

$$\left\{ \begin{array}{ll} \xi_1' = \xi_1, & \xi_i' = \sum_{j=2}^{m} a_{ij}\xi_j \\ \eta_1' = \eta_1, & \eta_i' = \sum_{j=2}^{m} \delta_{ij}\eta_j \end{array} \right\} \quad (j = 2, \ldots, m).$$

On the Influence of Gravity on Elastic Waves, and, in particular, on the Vibrations of an Elastic Globe. By T. J. I'A. BROMWICH, B.A., Fellow of St. John's College, Cambridge.

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This paper contains solutions of four elastic problems, all originally suggested by hypotheses which might modify the velocity of propagation of shocks along the surface of the earth. The first, second, and third deal with gravitational effects; hence, in these three I have assumed the material incompressible in order to avoid the difficulties that arise, even in the statical problem, if the material be compressible (Love's *Elasticity*, Vol. I., Art. 127). In the fourth

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* Dickson, *Annals of Mathematics*, pp. 161–183, 1897.

problem I consider the effect of a thin skin, whose elastic constants differ from those of the main body; here gravity does not enter into the problem and the material is supposed compressible.

The first, second, and fourth cases suppose the free surface to be an infinite plane: these are based on a paper by Lord Rayleigh (*Proc. Lond. Math. Soc.*, Vol. XVII.). From the first and second it appears that when the length of waves is short enough for us to regard the earth as plane the effect of gravity must be in all cases small. The fourth solution shows that the effect of the skin must be proportional to its thickness, and hence must be small.

The third problem solves the vibration of a sphere under its own gravity. Here the modification introduced by gravity appears to be considerable, on using the approximate elastic constants of the earth. The method adopted here is practically the same as Prof. Lamb's, to be found in his well-known paper on a vibrating sphere (*Proc. Lond. Math. Soc.*, Vol. XIII.); but I have used a slightly modified form for the analysis, which reduces the labour of manipulation and also gives a more convenient form of the period-equation.

It appears that gravity has no effect if the order of the harmonic disturbance is zero or unity; when this order is 2, I have calculated a number of roots of the period-equation. In particular for a sphere of the size, mass, and gravity of the earth, but with rigidity about that of steel, the gravest free period is 55 minutes; the corresponding period without gravity is 66 minutes. If the rigidity be about that of glass, the periods are 78 and 120 minutes, respectively.

These problems were originally undertaken at the suggestion of Dr. Larmor, to whom I am indebted for many valuable criticisms.

1. *Propagation of Waves under Constant Gravity on the Surface of an Infinite Incompressible Elastic Solid with an Infinite Horizontal Face.*

Following Lord Rayleigh's method (*loc. cit. supra*), we have to make but one modification, viz., the normal traction on the mean free surface has to be just sufficient to support the weight of the harmonic inequality, instead of vanishing. The proof of this statement will be found in Love's *Elasticity* (Vol. I., Art. 173).

To shorten the work, I take the axis of x to be the direction of propagation of the waves; then, if z is vertically upwards, we take all the displacements independent of y and $v=0$. The ordinary

equations of elasticity then become

$$\rho\frac{\partial^2 u}{\partial t^2} = (\lambda+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 u,$$

$$\rho\frac{\partial^2 w}{\partial t^2} = (\lambda+\mu)\frac{\partial\Delta}{\partial z}+\mu\nabla^2 w,$$

$$\Delta = \frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}.$$

But, since the solid is incompressible, Δ, the dilatation, will be zero; however, $\lambda\Delta$ will be finite, and let us put $p_1 = \lambda\Delta$, so that p_1 is a kind of negative hydrostatic pressure. We then have the modified equations

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial p_1}{\partial x}+\mu\nabla^2 u,$$

$$\rho\frac{\partial^2 w}{\partial t^2} = \frac{\partial p_1}{\partial z}+\mu\nabla^2 w,$$

$$0 = \frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}.$$

Now assume all the displacements to contain the factor $\exp(ipt+ilx)$; so that the wave-length $\lambda' = 2\pi/l$, and the velocity of propagation is l/p. We then have, if

$$\kappa^2 = \rho p^2/\mu,$$

$$(\nabla^2+\kappa^2)\,u = -\frac{1}{\mu}\frac{\partial p_1}{\partial x},$$

$$(\nabla^2+\kappa^2)\,w = -\frac{1}{\mu}\frac{\partial p_1}{\partial z},$$

whence we find $$\nabla^2 p_1 = 0.$$

Thus we take $$p_1/\mu\kappa^2 = (Pe^{-lz}+Qe^{lz})\exp(ipt+ilx),$$

which is the general solution, if p_1 contains the exponential $\exp(ipt+ilx)$. Now in the solid z ranges from 0 at the mean free surface to $-\infty$; consequently, if l be positive, we must take $P = 0$, so that p_1 may not increase indefinitely with the depth. Whence

$$p_1/\mu\kappa^2 = Qe^{lz}\exp(ipt+ilx).$$

Next, since $\nabla^2 p_1 = 0$, a particular set of values of the displacements will be

$$u = -\frac{1}{\mu\kappa^2}\frac{\partial p_1}{\partial x},$$

$$w = -\frac{1}{\mu\kappa^2}\frac{\partial p_1}{\partial z},$$

to which we must add complementary solutions of the equations

$$(\nabla^2+\kappa^2)\, u = 0,$$

$$(\nabla^2+\kappa^2)\, w = 0,$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z} = 0,$$

so that in all we find

$$u = -\frac{1}{\mu\kappa^2}\frac{\partial p_1}{\partial x} + Ae^{sz}\exp i\,(lx+pt),$$

$$w = -\frac{1}{\mu\kappa^2}\frac{\partial p_1}{\partial z} + Be^{sz}\exp i\,(lx+pt),$$

where $\quad ilA+sB = 0 \quad$ and $\quad s^2+\kappa^2 = l^2.$

It is assumed that the real part of s is positive in order that u, w may not become infinite at $z = -\infty$; the case when s is purely imaginary will be considered later.

The conditions at $z = 0$ are

$$\lambda\Delta + 2\mu\frac{\partial w}{\partial z} + g\rho w = 0,$$

$$\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) = 0,$$

the first of which makes the normal traction on the mean free surface just support the weight of the harmonic inequality, while the second makes the tangential traction zero. Whence

$$\mu\kappa^2 Q + 2\mu\,(sB - l^2Q) + g\rho\,(B - lQ) = 0,$$

$$-2il^2Q + sA + ilB = 0.$$

But $ilA = -sB$, and thus we have

$$-lQ\,(2-\kappa^2/l^2) + (g\rho/\mu l)(B-lQ) + 2sB/l = 0,$$

$$2lQ - (2-\kappa^2/l^2)\,B = 0,$$

in the second of which we have put

$$s^2 = l^2 - \kappa^2.$$

Writing now $\kappa^2/l^2 \equiv \zeta$, we find, after eliminating the ratio $B : Q$,

$$(2-\zeta)(2-\zeta+g\rho/\mu l) = 2\,(2\sqrt{1-\zeta}+g\rho/\mu l),$$

or

$$(2-\zeta)^2 - 4\sqrt{1-\zeta} - \zeta\,(g\rho/\mu l) = 0,$$

and here $\sqrt{1-\zeta} = s/l$, and so the real part of $\sqrt{1-\zeta}$ is to be positive.

When $g = 0$, the equation is the same as that found by Rayleigh for an incompressible solid (*loc. cit. supra*).

I now proceed to obtain an estimate of the magnitude of $(g\rho/\mu l)$. In the fifteenth *Brit. Assoc. Report*, "On the Earthquake Phenomena of Japan" (p. 58), we find that approximate values near the earth's surface are, in C.G.S. units,

$$\rho = 3, \quad \mu = (1{\cdot}5)\,10^{11},$$

and the mean value of g is known to be 981 in these units. Now

$$g\rho/\mu l = g\rho\lambda'/2\pi\mu,$$

and with the values above we find roughly

$$2\pi\mu/g\rho = (3{\cdot}\dot{2}04)\,10^8;$$

also in centimetres the earth's mean radius $a = (6{\cdot}37)\,10^8$ nearly. Thus a rough estimate of $g\rho/\mu l$ is $2\lambda'/a$.

Now it is clear that λ'/a must be small in order that we may treat the earth as approximately plane. Consequently the roots of my period-equation cannot differ greatly from those given by Rayleigh. Suppose, then, ζ_0 to be a root of Rayleigh's equation, and now put

$$\zeta = \zeta_0 + \delta\zeta;$$

we have then, approximately,

$$2\delta\zeta\left[(1-\zeta_0)^{-\frac{1}{2}} - (2-\zeta_0)\right] - (g\rho/\mu l)\,\zeta_0 = 0;$$

but

$$4\,(1-\zeta_0)^{\frac{1}{2}} = (2-\zeta_0)^2,$$

so that this becomes

$$2\,(\delta\zeta/\zeta_0)\left[4\,(2-\zeta_0)^{-2} - (2-\zeta_0)\right] = g\rho/\mu l.$$

Lord Rayleigh shows that, of the three values of ζ_0 which differ from zero, only one is a solution of the problem, as the other two make the real part of $\sqrt{1-\zeta_0}\ (= s/l)$ negative, which is inadmissible.

This value of ζ_0 is given by him as 0·91275, but my calculations have led me to 0·91262. Taking this value, we have

$$4\,(2-\zeta_0)^{-2}-(2-\zeta_0) = 2{\cdot}2956 \text{ nearly,}$$

and so $$\delta\zeta/\zeta_0 = (0{\cdot}2178)(g\rho/\mu l) \text{ nearly.}$$

The velocity of propagation $V = p/l = (\mu\zeta/\rho)^{\frac{1}{2}}$; thus, if $V_0 = (\mu\zeta_0/\rho)^{\frac{1}{2}}$, we have, to the same degree of accuracy as before,

$$(V-V_0)/V_0 = \delta\zeta/2\zeta_0 = (0{\cdot}1089)(g\rho/\mu l) = (0{\cdot}213)(\lambda'/a),$$

with the values of μ, ρ quoted above. Consequently the ratio $(V-V_0)/V_0$ must be a very small fraction in all cases to which this method of approximation can be applied.

After the above solution had been completed, it was pointed out as a means of verification that the period-equation ought to lead to the known value of the velocity of propagation of short waves on water. Making μ small in the equation, I found as the first approximation to ζ the value $g\rho/\mu l$, giving the velocity $(\zeta\mu/\rho)^{\frac{1}{2}} = (g/l)^{\frac{1}{2}} = (g\lambda'/2\pi)^{\frac{1}{2}}$, the well-known form. But this would clearly make $(1-\zeta)^{\frac{1}{2}}$ imaginary, and thus the terms neglected in ζ would have to be complex, leading to a complex period. Since this is inadmissible, it will be advisable to examine the assumptions made above.

It now appears that when s is purely imaginary the values of u, w may include terms in e^{-sz} as well as those in e^{sz}, both sets being finite at $z = -\infty$. This will introduce a new arbitrary constant; and hence also an additional boundary-condition. To express this condition in the simplest way, take the solid as a slab of thickness $2h_0$, where h_0 will be subsequently made infinite. I shall replace the terms in e^{lz}, e^{sz}, &c., by hyperbolic functions, and take the origin as midway between the two faces of the slab. Thus we have

$$p_1/\mu\kappa^2 = A\cosh(lz) + B\sinh(lz),$$

$$u = -\frac{1}{\mu\kappa^2}\frac{\partial p_1}{\partial x} + X_1\cosh(sz) + X_2\sinh(sz),$$

$$w = -\frac{1}{\mu\kappa^2}\frac{\partial p_1}{\partial z} + Z_1\cosh(sz) + Z_2\sinh(sz).$$

Also $$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

so we find $$ilX_1 + sZ_2 = 0 \quad\text{and}\quad ilX_2 + sZ_1 = 0,$$

the factor $\exp(ipt + ilx)$ in p_1, u, w having been suppressed for brevity.

I take as the boundary conditions

$$p_1+2\mu\frac{\partial w}{\partial z}+g\rho w=0,$$

$$\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=0,$$

at each mean free surface $z=h_0$ and $z=-h_0$. Thus we have

$$(\kappa^2-2l^2)\left[A\cosh(lh_0)+B\sinh(lh_0)\right]+2s\left[Z_1\sinh(sh_0)+Z_2\cosh(sh_0)\right]$$
$$+(g\rho/\mu)\left[Z_1\cosh(sh_0)+Z_2\sinh(sh_0)\right.$$
$$\left.-l\left\{A\sinh(lh_0)+B\cosh(lh_0)\right\}\right]=0,$$

from the first condition; and, from the second,

$$-2il^2\left[A\sinh(lh_0)+B\cosh(lh_0)\right]+s\left[X_1\sinh(sh_0)+X_2\cosh(sh_0)\right]$$
$$+il\left[Z_1\cosh(sh_0)+Z_2\sinh(sh_0)\right]=0,$$

together with two similar equations which are the same as these when the sign of h_0 is changed. Substituting in the second of these for X_1, X_2 in terms of Z_1, Z_2, we have

$$-2l^3\left[A\sinh(lh_0)+B\cosh(lh_0)\right]$$
$$+(2l^2-\kappa^2)\left[Z_1\cosh(sh_0)+Z_2\sinh(sh_0)\right]=0.$$

Whence, changing the sign of h_0 and adding and subtracting, we have

$$(\kappa^2-2l^2)A\cosh(lh_0)+2sZ_2\cosh(sh_0)$$
$$+(g\rho/\mu)\left[Z_1\cosh(sh_0)-lB\cosh(lh_0)\right]=0,$$

$$(\kappa^2-2l^2)B\sinh(lh_0)+2sZ_1\sinh(sh_0)$$
$$+(g\rho/\mu)\left[Z_2\sinh(sh_0)-lA\sinh(lh_0)\right]=0,$$

$$(2l^2-\kappa^2)Z_1\cosh(sh_0)-2l^3B\cosh(lh_0)=0,$$
$$(2l^2-\kappa^2)Z_2\sinh(sh_0)-2l^3A\sinh(lh_0)=0;$$

whence we find, eliminating Z_1, Z_2,

$$-(2l^2-\kappa^2)^2A\coth(lh_0)+4l^3sA\coth(sh_0)+(g\rho/\mu)\,l\kappa^2B\coth(lh_0)=0,$$
$$-(2l^2-\kappa^2)^2B\tanh(lh_0)+4l^3sB\tanh(sh_0)+(g\rho/\mu)\,l\kappa^2A\tanh(lh_0)=0.$$

Eliminating the ratio $A:B$, we have*

$$\left[(2l^2-\kappa^2)^2\tanh(lh_0)-4l^3s\tanh(sh_0)\right]$$
$$\times\left[(2l^2-\kappa^2)^2\coth(lh_0)-4l^3s\coth(sh_0)\right]=(g\rho l\kappa^2/\mu)^2.$$

* On putting $g=0$ this reduces to two period-equations which agree with (38), (47) of Lord Rayleigh's paper "On the Vibrations of an Infinite Plate" (*Proc. Lond. Math. Soc.*, Vol. xx.).

Now, consider the limiting form of this period-equation when h_0 is indefinitely increased; coth (lh_0) approaches the limit unity, so also does coth (sh_0), provided that the real part of s is positive. This was the condition previously imposed on s; we shall hold over for the moment the consideration of the case when s is purely imaginary. Our period-equation is thus

$$\left[(2l^2-\kappa^2)^2-4l^3s\right]^2 = (g\rho l\kappa^2/\mu)^2,$$

and it will be seen that we must choose

$$(2l^2-\kappa^2)^2 = 4l^3s+g\rho l\kappa^2/\mu,$$

in order that $z = +h_0$ may be the surface at which the disturbance is finite. This equation is the same as that found previously.

Next take $s = is'$, where s' is supposed real. Then

$$s \tanh(sh_0) = -s' \tan(s'h_0), \quad \text{and} \quad s \coth(sh_0) = s' \cot(s'h_0);$$

these two expressions do not tend to limits independent of s' as h_0 is increased indefinitely. Thus here the period-equation must involve h_0; but we can obtain an approximate solution when μ is small. In this case κ will be large, provided p, ρ be supposed finite. Our equation will then yield approximately

$$(\kappa^4)^2 = (g\rho l\kappa^2/\mu)^2;$$

whence $$\kappa^2 = g\rho l/\mu, \quad \text{or} \quad p^2 = gl,$$

which gives the velocity of wave propagation

$$p/l = (g/l)^{\frac{1}{2}} = (g\lambda'/2\pi)^{\frac{1}{2}}.$$

This is the well-known result for the velocity of propagation on water of waves whose length is short compared with the depth.

It will be noticed that the equation originally found,

$$(2l^2-\kappa^2)^2 = 4l^3s+g\rho\kappa^2 l/\mu,$$

always gives a real value of (κ^2/l^2) which lies between 0 and 1. Apparently we should thus have in all cases a real value of s given by this equation; but when the ratio $(g/l) : (\mu/\rho)$ is greater than unity it will be found that this value of s must be negative in order to satisfy the period-equation, and this must be excluded according to the original conditions. Hence, if $(g/l) > (\mu/\rho)$, *i.e.*, if the velocity of propagation due to gravity alone be greater than that of rotational waves, then the more complicated period-equation just found must be used.

It will be observed that in the physical application originally considered $g\rho/\mu l$ was a small fraction, and consequently this point did not present itself.

2. *The effect on the previous problem due to an Ocean of Depth small compared with the Wave-length.*

For simplicity take the depth as uniform, so that the mean boundaries are two infinite horizontal planes. Neglecting viscosity, the motion in the water is irrotational; let ϕ be the velocity-potential with $\partial\phi/\partial s$ as the velocity in the direction ds.

Retaining the axes and notation of the former problem, we write at once, in the solid,

$$p_1/\mu\kappa^2 = Qe^{lz}\exp(ipt+ilx),$$

$$u = -\frac{1}{\mu\kappa^2}\frac{\partial p_1}{\partial x} + Ae^{sz}\exp(ipt+ilx),$$

$$w = -\frac{1}{\mu\kappa^2}\frac{\partial p_1}{\partial z} + Be^{sz}\exp(ipt+ilx),$$

where $$ilA+sB=0 \quad\text{and}\quad s^2+\kappa^2-l^2=0.$$

To determine ϕ we have $\nabla^2\phi=0$, and hence

$$\phi = \left[C\cosh(lz)+D\sinh(lz)\right]\exp(ipt+ilx).$$

Next we have at $z=0$ $$\frac{\partial\phi}{\partial z} = \frac{\partial w}{\partial t},$$

which gives $$lD = ip(B-lQ).$$

At the free surface ($z=h_0$, when undisturbed) the pressure must be constant. Thus

$$g\frac{\partial\phi}{\partial z}+\frac{\partial^2\phi}{\partial t^2} = 0 \quad\text{at}\quad z=h_0,$$

or

$$gl\left[C\sinh(lh_0)+D\cosh(lh_0)\right]-p^2\left[C\cosh(lh_0)+D\sinh(lh_0)\right]=0.$$

Now $lh_0\ (=2\pi h_0/\lambda')$ is supposed to be small; so, approximately,

$$\sinh(lh_0)=lh_0 \quad\text{and}\quad \cosh(lh_0)=1;$$

whence $$gl(D+Clh_0)-p^2(C+Dlh_0)=0.$$

At $z = 0$ we have the two conditions

$$\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=0,$$

$$p_1+2\mu\frac{\partial w}{\partial z}+g(\rho-\rho')w-\rho'\frac{\partial\phi}{\partial t}=0,$$

where, in the second condition, ρ' is the density of the water; and the effect of the water-pressure has been included. Thus

$$-2il^2Q+sA+ilB=0,$$

$$\mu\kappa^2Q+2\mu(-l^2Q+sB)+g(\rho-\rho')(-lQ+B)-ip\rho'C=0.$$

Now we have

$$C(p^2-gl^2h_0)=Dl(g-p^2h_0)=ip(g-p^2h_0)(B-lQ),$$

which gives, approximately,

$$pC=i(B-lQ)(g-p^2h_0),$$

the terms rejected being of order (gl^2h_0/p^2) in comparison with those retained. Now (gl^2h_0/p^2) is $(g\rho/\mu l)(lh_0)(l^2/\kappa^2)$, and, by what has been already proved in the first section, $(g\rho/\mu l)$ is a small fraction, while lh_0 is also small. We thus have, on substituting for A and C in terms of B, Q,

$$2l^3Q-(s^2+l^2)B=0$$

and

$$\mu(\kappa^2-2l^2)Q+2\mu sB+(g\rho-\rho'p^2h_0)(B-lQ)=0.$$

These give

$$(2-\kappa^2/l^2)^2-4s/l-(g\rho/\mu l)(\kappa^2/l^2)+(\rho'/\rho)(lh_0)(\kappa^4/l^4)=0.$$

Writing $\zeta=\kappa^2/l^2$ as before, this becomes

$$(2-\zeta)^2=4\sqrt{1-\zeta}+(g\rho/\mu l)\zeta-(lh_0\rho'/\rho)\zeta^2.$$

Obviously, if $\rho'/\rho=0$, or if $lh_0=0$, we get back to the period-equation found in the first section. Solving by approximation in the same way, we get

$$2(\delta\zeta/\zeta_0)\left[4(2-\zeta_0)^{-2}-(2-\zeta_0)\right]=(g\rho/\mu l)-(lh_0\rho'/\rho)\zeta_0,$$

which yields with $\zeta_0=0{\cdot}91262$

$$(V-V_0)/V_0=\delta\zeta/2\zeta_0=(0{\cdot}109)(g\rho/\mu l)-(0{\cdot}099)(lh_0\rho'/\rho).$$

Expressed in terms of the wave-length, with the same values of μ, ρ as used above,

$$(V-V_0)/V_0=(0{\cdot}213)(\lambda'/a)-(0{\cdot}522)(\rho'/\rho)(h_0/\lambda').$$

3. *The Vibrations of an Incompressible Sphere under its own Gravity.*

We shall neglect the central part of gravity in solving for u, v, w, as its only effect is to introduce into the traction on the mean free surface a term which is equal to the weight of the harmonic inequality (Love's *Elasticity*, Vol. I., Art. 173). But we must retain the gravitational potential of the harmonic inequality, which we denote by V, so that V contains terms of the same order as the displacements.

We then have the differential equations of motion

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial p_1}{\partial x} + \mu\nabla^2 u + \rho\frac{\partial V}{\partial x},$$

$$\rho\frac{\partial^2 v}{\partial t^2} = \frac{\partial p_1}{\partial y} + \mu\nabla^2 v + \rho\frac{\partial V}{\partial y},$$

$$\rho\frac{\partial^2 w}{\partial t^2} = \frac{\partial p_1}{\partial z} + \mu\nabla^2 w + \rho\frac{\partial V}{\partial z},$$

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},$$

the last equation holding on account of the incompressibility, and in the others $\lambda\Delta = p_1$, a finite quantity. It at once appears that

$$\nabla^2 p_1 = 0 \quad \text{since} \quad \nabla^2 V = 0$$

by properties of the potential.

We thus get a set of particular integrals

$$(u_1, v_1, w_1) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi,$$

where $\qquad \phi \equiv -(p_1 + \rho V)/\mu\kappa^2,$

and, as before, $\qquad \kappa^2 = \rho p^2/\mu.$

The complementary solutions are to satisfy

$$(\nabla^2 + \kappa^2)\, u_2 = 0,$$

$$(\nabla^2 + \kappa^2)\, v_2 = 0,$$

$$(\nabla^2 + \kappa^2)\, w_2 = 0,$$

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial w_2}{\partial z} = 0.$$

I shall now introduce the hypothesis that the displacements are symmetrical round an axis; this is really perfectly general, for by superposition of such solutions we can get every possible case. We reject the displacements called by Prof. Lamb "those of the first class," in which the displacement is in circles round the axis; and proceed at once to those of the second class, where the displacement is in a meridian plane. In displacements of the first class there is no radial motion; consequently the effect of gravity is nil.

For the future u, v will represent the radial and transverse displacements in the directions of r, θ respectively increasing; the notation is that of three-dimensional polars. Then

$$u = \frac{\partial\phi}{\partial r} - \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta},$$

$$v = \frac{1}{r}\frac{\partial\phi}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r},$$

where

$$\phi = -(p_1+\rho V)/\mu\kappa^2,$$

as before; and the terms in ψ give the complementary solutions u_2, v_2, w_2 of the previous notation. Here ψ satisfies

$$(D+\kappa^2)\psi = 0,$$

where

$$D \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right),$$

which is the operator usually associated with a Stokes' stream-function.

We then find that typical terms in ϕ, ψ are, if P_n is Legendre's coefficient of degree n in $\cos\theta$,

$$\phi = Ar^nP_ne^{ipt},$$

$$\psi = Br^{n+1}\psi_n(\kappa r)\sin\theta\frac{dP_n}{d\theta}e^{ipt},$$

where

$$\psi_n(x) \equiv \left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right)$$

$$\equiv \frac{1}{1.3.5\ldots 2n+1}\left[1 - \frac{x^2}{2.2n+3} + \frac{x^4}{2.4.2n+3.2n+5} - \ldots\right],$$

according to the notation of Prof. Lamb (Hydrodynamics, Art. 287).

This value of ψ is at once obvious by remembering that

$$D\psi \equiv \frac{r\sin\theta}{\cos\omega}\nabla^2\left(\frac{\psi\cos\omega}{r\sin\theta}\right),$$

ω being the azimuthal angle of polars.

We then have

$$u = nr^{n-1}\left[A+(n+1)B\psi_n(\kappa r)\right]P_n e^{ipt},$$

$$v = r^{n-1}\left[A+\left\{(n+1)\psi_n(\kappa r)+\kappa r\psi_n'(\kappa r)\right\}B\right]\frac{dP_n}{d\theta}e^{ipt}.$$

From this value of u we see at once that V is of the form

$$3gu_0 r^n/(2n+1)a^n,$$

where u_0 is the value of u at $r=a$. Also p_1 satisfies $\nabla^2 p_1 = 0$, and so we put $p_1 = \beta u_0 r^n/a^n$, where β is a constant. To determine β, we have

$$\mu\kappa^2\phi = -(p_1+\rho V),$$

and hence $\mu\kappa^2 Aa + n\left[\beta+3g\rho/(2n+1)\right]\left[A+(n+1)B\psi_n(\kappa a)\right] = 0.$

The equations to be satisfied at the surface are now seen to be

$$p_1 + 2\mu\frac{\partial u}{\partial r} + g\rho u = 0,$$

and
$$r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial\theta} = 0.$$

Substituting, we get the two conditions

$$n\left[\beta+g\rho+2\mu(n-1)/a\right]\left[A+(n+1)B\psi_n(\kappa a)\right] + 2\mu\kappa n(n+1)B\psi_n'(\kappa a) = 0$$

and $2(n-1)A + B\left[2(n^2-1)\psi_n(\kappa a) - 2\kappa a\psi_n'(\kappa a) - \kappa^2 a^2\psi_n(\kappa a)\right] = 0,$

in the second of which $\psi_n''(\kappa a)$ has been expressed by $\psi_n(\kappa a)$ and $\psi_n'(\kappa a)$. Substituting for β in the first of these, we find

$$2n(n-1)\left[1+g\rho a/\mu(2n+1)\right]\left[A+(n+1)B\psi_n(\kappa a)\right] - \kappa^2 a^2 A + 2\kappa an(n+1)B\psi_n'(\kappa a) = 0.$$

For brevity put

$$B\psi_n(\kappa a)\equiv C,\quad \kappa a\equiv x,\quad \psi_n'(\kappa a)/\psi_n(\kappa a)\equiv X,\quad ng\rho a/(2n+1)\mu\equiv\theta;$$

then we have

$$2(n-1)(n+\theta)\left[A+(n+1)C\right] - x^2 A + 2n(n+1)XxC = 0,$$

while the second surface-condition becomes

$$2(n-1)A + C\left[2(n^2-1) - 2Xx - x^2\right] = 0.$$

On eliminating the ratio $A : C$ and rejecting some superfluous factors, these give

$$(n+1)(x+2nX) + \left[n+\theta - x^2/2(n-1)\right](x+2X) = 0,$$

which may be written

$$\frac{2}{\kappa a}\frac{\psi_n'(\kappa a)}{\psi_n(\kappa a)} + \frac{(2n+1) + ng\rho a/(2n+1)\,\mu - \kappa^2a^2/2(n-1)}{n(n+2) + ng\rho a/(2n+1)\,\mu - \kappa^2a^2/2(n-1)} = 0.$$

By putting $g = 0$ we arrive at an equation which is the same as that found by Prof. Lamb (*Proc. Lond. Math. Soc.*, Vol. XIII.), when allowance is made for the fact that the value of $\psi_n(\kappa a)$ which is there adopted is $[1.3.5 \ldots 2n+1]$ times the value used above.

From the form of the period-equation above it appears that $n = 0$, $n = 1$ define modes of vibration which are not affected by gravity.

It is of interest to see that the equation just found reduces to the form given previously when we considered an infinite solid with a plane face. We take a, n as both infinite and the harmonics as sectorials; then $2\pi a/n$ = wave-length = $2\pi/l$ of former work; so $n = al$. We must now investigate the form of ψ_n when both n and the argument are very great. I have not succeeded in finding a known form either of ψ_n or of $J_{n+\frac{1}{2}}$ in this case; accordingly I proceed to determine a form by first principles. We have here that, with $\theta = \pi/2$, $(r^n\psi_n)\,e^{in\omega} = U$ is a solution of

$$(\nabla^2+\kappa^2)\,U = 0 \quad \text{and} \quad n\omega = la\omega = lx;$$

so we have $$r^n\psi_n = Ae^{s(r-a)},$$

where $$s^2 = \kappa^2 + l^2,$$

and the real part of s is positive, so that ψ_n may not be infinite at $r = 0$.*

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* Another method is as follows:—$\psi_n(\kappa r)$ is a solution of

$$\frac{d^2y}{dr^2} + \frac{2(n+1)}{r}\frac{dy}{dr} + \kappa^2 y = 0.$$

Now write $r = a - z$, and suppose z/a to be small; the equation for y will become

$$\frac{d^2y}{dz^2} - 2l\frac{dy}{dz} + \kappa^2 y = 0,$$

and we find $$\psi_n = Ae^{(l-s)z}.$$

Differentiate now with respect to r and put $r = a$; we find

$$\frac{n}{a} + \frac{\kappa\psi_n'(\kappa a)}{\psi_n(\kappa a)} = s,$$

i.e.,

$$\frac{\psi_n'(\kappa a)}{\psi_n(\kappa a)} = \frac{s-l}{\kappa}.$$

Also $$ng\rho a/(2n+1)\,\mu = ng\rho/2\mu l$$

in the limit; thus the period-equation becomes

$$2\,(s-l)/\kappa^2 a + (2+g\rho/2\mu l - \kappa^2/2l^2)/n = 0,$$

which is equivalent to

$$4l\,(s-l) + \kappa^2\,(4+g\rho/\mu l - \kappa^2/l^2) = 0,$$

and with $$\zeta = \kappa^2/l^2,$$

as before, we find $(2-\zeta)^2 = 4\sqrt{1-\zeta} + (g\rho/\mu l)\,\zeta$,

the form already given in Section 1.

An additional verification is afforded by taking μ extremely small; we ought then to find one of the periods the same as that given by Kelvin's formula for a gravitating fluid sphere (*Phil. Trans.*, 1863).

Taking μ as very small, p^2 being kept finite, κ will be very great, and then, after multiplying up, the most important terms in the period-equation contain the factor

$$ng\rho a/(2n+1)\,\mu - \rho p^2 a^2/2\,(n-1)\,\mu,$$

and thus the approximate period-equation may be taken as

$$p^2 = 2n\,(n-1)\,g/(2n+1)\,a,$$

which is Kelvin's formula.

I now proceed to the discussion of the roots of the period-equation. We see that $n = 2$ is the first harmonic which gives any difference from the case without gravity; and for the future this alone will be considered. The equation is

$$\frac{2}{x}\,\frac{\psi_2'(x)}{\psi_2(x)} + \frac{5+2\gamma/5-x^2/2}{8+2\gamma/5-x^2/2} = 0,$$

where γ denotes $g\rho a/\mu$. This can be reduced to the equivalent form

$$\frac{\tan x}{x} = \frac{24\,(20+\gamma) - 4\,(23+\gamma)\,x^2 + 5x^4}{24\,(20+\gamma) - 12\,(21+\gamma)\,x^2 + (25+4\gamma/5)\,x^4 - x^6},$$

remembering that

$$\psi_2(x) = \left[(3-x^2)\sin x - 3x\cos x\right]/x^5.$$

The second form will be seen to reduce to equation (80) of Prof. Lamb's paper previously quoted, on putting $\gamma = 0$.

I originally attempted to solve the equation by assuming a value of γ, and then using the method of trial and error. By this means I calculated the roots marked (A) in the table subjoined. But it soon became clear that, to trace the roots systematically, an easier plan would be to evaluate the values of γ corresponding to assumed values of x. To do this I tabulated $\psi_2(x)$, $\psi_2'(x)$, and deduced the values of $2\psi_2'(x)/x\psi_2(x)$ corresponding to values of x, differing by $\pi/10$. The calculation of γ then offers but little difficulty. The periods were deduced for a sphere of the same size as that of the earth, with the same surface-value of gravity, using the constants

$$a = (6{\cdot}37)\,10^8, \quad g = (9{\cdot}80)\,10^2.$$

I proceed to make a few notes on my results. Taking μ about the rigidity of steel, I calculate that $\gamma = 4{\cdot}32$, which gives a period about 55 minutes, as against 66 minutes found by neglecting gravity; and, with μ about the rigidity of glass, $\gamma = 15$ nearly, which gives the gravest period about 78·5 minutes, as against 120 minutes when gravity is neglected. These are the cases of chief physical interest.

A general description of the variation of the roots with γ may make the table clearer. The lowest root is $(\cdot 8485)\,\pi$ when $\gamma = 0$, according to Prof. Lamb; this root increases with γ, until γ becomes ∞, corresponding to a value of x between $(1{\cdot}65)\,\pi$ and $(1{\cdot}70)\,\pi$, the period at the same time increasing to ∞. After this, until $x = (1{\cdot}7420)\,\pi$, the value of γ is increasing from $-\infty$ to 0, which indicates that these values of x cannot occur in any real case. We now come to a series of second roots of the period-equation; here the value of γ at first varies rapidly for small variations of x, and for a value of x between $(2{\cdot}8)\,\pi$ and $(2{\cdot}8257)\,\pi$ becomes ∞; it then changes very rapidly from $-\infty$ to 0. The third, fourth, and fifth roots have the same general properties, but it is remarkable that, as the order of the root increases, so also does the value of γ requisite to produce a given period. Moreover it appears that, in the higher periods, the variation in the period is slight in comparison with the variation in γ; also, as the order of the period increases, so does the range of values of γ for which the period differs but little from 94 minutes. It is in this sense that we must understand the period 94 minutes, as found by Kelvin's result for a gravitating fluid sphere of the same size and gravity as the earth. Of course every value of γ gives rise to an infinity of periods, and the particular case of $\gamma = \infty$, corre[illegible]

to a fluid sphere, gives an infinity of infinite periods and a finite period 94 minutes.

The table contains about two-thirds of the periods I have calculated, those not inserted can be interpolated with sufficient accuracy.

TABLE of Periods of a Gravitating Elastic Sphere, with the same Radius and Surface-Gravity as the Earth, tabulated for varying values of $(g\rho a/\mu)$:—

	$g\rho a/\mu$.	$\kappa a/\pi$.	Period in Minutes.			$g\rho a/\mu$.	$\kappa a/\pi$.	Period in Minutes.	
L	0	0·8485	0	Lowest roots of the period-equation	(3) L	0	2·8257	0	Third roots
	3·8	1·0	52·5			84·5	2·9	85	
A (1)	4·32	1·019	55			96	3·0	88	
	6·8	1·1	63 5			173	3·8	93	
	10·9	1·2	74			193·5	3·85	97	
	13·9	1·3	77						
(2)	18·0	1·4	81·5		(3) L	0	3·8709	0	Fourth
	24·6	1·5	89			153	3·9	85	
	36·5	1·6	102			274	4·8	93	
	56·2	1·65	141						
(3) L	0	1·7420	0	Second roots	(3) L	0	4·8974	0	Fifth
A	15	1·794	58			198	4·9	77	
	27·3	1·9	74			436	5·9	95	
	40·9	2·1	82		(3) L	0	5·9148	0	Sixth
	53·2	2·3	85·5			425	6·0	92	
	66·5	2·5	88						
	84·0	2·7	91						
	118·0	2·8	104						

REMARKS.

L indicates that the root is taken from Prof. Lamb's paper.

A these roots were found by a different method from the rest.

(1) μ = that of steel.

(2) $(g\rho a/\mu) = 15$ nearly, if μ be that of glass, so the corresponding period is about 78 minutes.

(3) in each of these intervals, the ratio $(g\rho a/\mu)$ changes very rapidly to $+\infty$, $-\infty$, and zero. It thus appears that certain values of $(\kappa a/\pi)$ cannot appear in the solution of this problem, viz., those which make $(g\rho a/\mu)$ negative. For instance, I find that $(\kappa a/\pi) = 1 \cdot 7$ makes this ratio negative, and so the values from $(1 \cdot 7)$ to $(1 \cdot 742)$ cannot appear.

Notation.—$2\pi/p$ = period, $\kappa^2 = \rho p^2/\mu$.

Three sets of curves are given to indicate graphically the results. In Fig. 1, the curves show the relation between $(g\rho a/\mu)$ and the *period*; they in all cases should go off to infinity, but owing to difficulties of computation it has not been possible to find the asymptotic

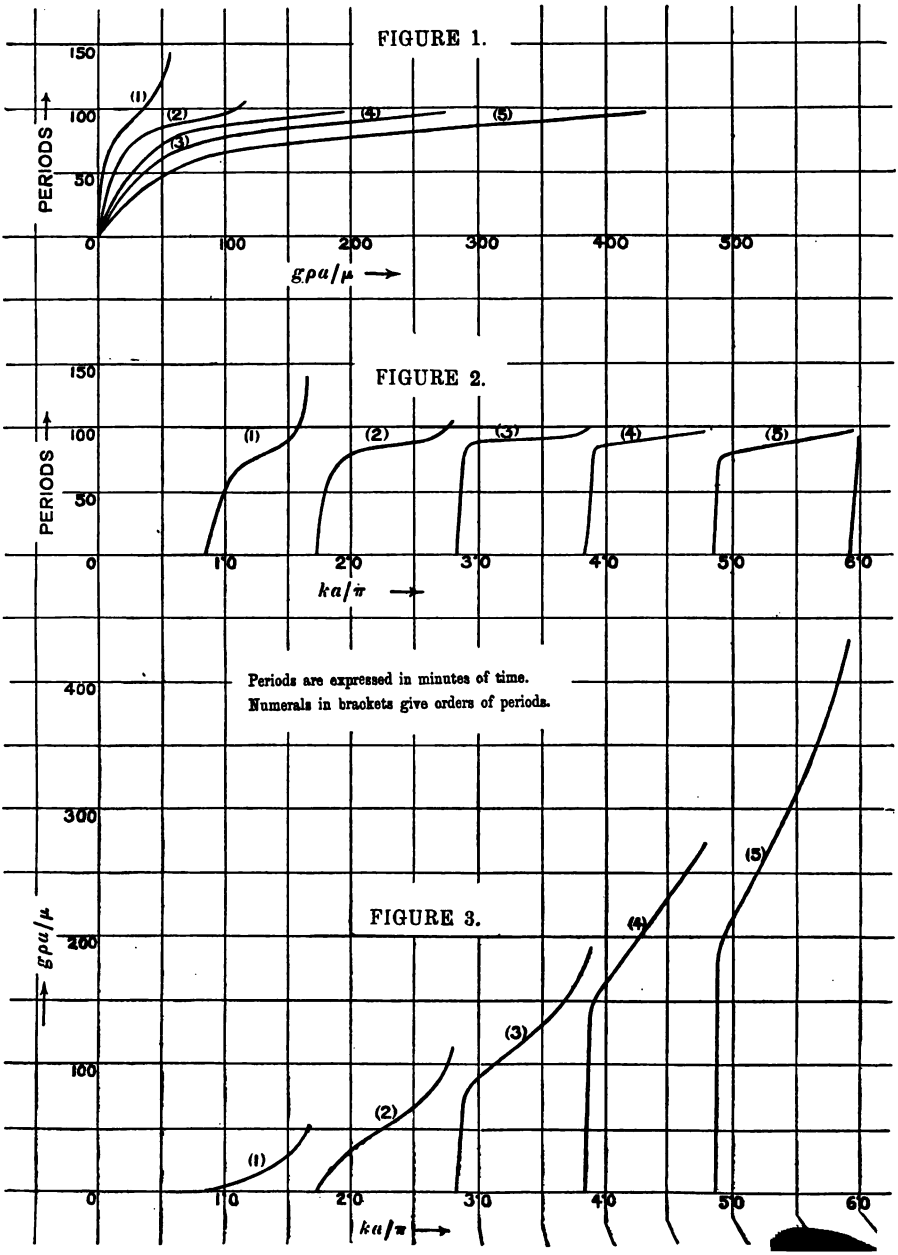

FIGURE 1.

FIGURE 2.

Periods are expressed in minutes of time.
Numerals in brackets give orders of periods.

FIGURE 3.

directions. Moreover curves (4) and (5) pass through the origin, but my calculations do not give the exact shape near the origin, which has been filled in by following the general outline of the first three.

In Figs. 2, 3, the abscissa is $(\kappa a/\pi)$, and the ordinates are the period and $(g\rho a/\mu)$ respectively. Here all the curves go off to positive infinity nearly vertically; and in Fig. 3 they return through negative infinity to the horizontal axis, in a nearly vertical direction. The negative part of the curves is not given, as it can have no physical interpretation, merely arising out of the analytical solutions.

4. *Propagation of Waves in a Thin Shell with Two Infinite Parallel Faces, one of which is rigidly attached to an Infinite Solid.*

The usual equations of small motion of an elastic solid in two dimensions are

$$\left.\begin{aligned} \rho\frac{\partial^2 u}{\partial t^2} &= (\lambda+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 u \\ \rho\frac{\partial^2 w}{\partial} &= (\lambda+\mu)\frac{\partial\Delta}{\partial z}+\mu\nabla^2 w \end{aligned}\right\},$$

where λ, μ are the elastic constants as defined in Love's *Elasticity*. I take the axis of x to be the direction of propagation of the waves, and that of z perpendicular to the plane boundaries, which will be the two planes $z=0$, $z=h_0$ in equilibrium. I suppose $z=h_0$ to be the free surface, and that h_0 is positive, so that $z=-\infty$ gives the other boundary of the infinite solid.

Also Δ is the dilation and is equal to

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}.$$

Now assume that u, w both contain the factor $\exp i\,(lx+pt)$; so that $2\pi/l$ is the wave-length, and p/l the velocity of propagation. Then, putting

$$h^2 = \rho p^2/(\lambda+2\mu) \quad\text{and}\quad \kappa^2 = \rho p^2/\mu,$$

the equations given above will reduce to

$$\left.\begin{aligned} (\nabla^2+\kappa^2)\,u &= \left(1-\frac{\kappa^2}{h^2}\right)\frac{\partial\Delta}{\partial x} \\ (\nabla^2+\kappa^2)\,w &= \left(1-\frac{\kappa^2}{h^2}\right)\frac{\partial\Delta}{\partial y} \end{aligned}\right\},$$

and so $$(\nabla^2+h^2)\,\Delta = 0.$$

Hence we assume that in the infinite solid

$$\Delta/h^2 = Ae^{rz},$$

$$u = -ilAe^{rz}+Xe^{sz},$$

$$w = -rAe^{rz}+Ze^{sz},$$

where $$r^2+h^2 = l^2 = s^2+\kappa^2,$$

and the real parts of r, s must be positive in order that u, w may vanish at $z = -\infty$; the exponential factor $\exp i\,(lx+pt)$ must be understood in all the terms on the right-hand side. From the value of Δ, we have at once $$ilX+sZ = 0.$$

Turning to the shell (whose elastic constants are supposed to be different, say λ', μ', ρ'), it will be seen that we are not restricted to one exponential in z, and for convenience I use two hyperbolic functions.

We may then write, for the displacements in the shell,

$$\Delta'/h'^2 = B\cosh(r'z)+C\sinh(r'z),$$

$$u' = -il\left[B\cosh(r'z)+C\sinh(r'z)\right]+X_1\cosh(s'z)+X_2\sinh(s'z),$$

$$w' = -r'\left[B\sinh(r'z)+C\cosh(r'z)\right]+Z_1\cosh(s'z)+Z_2\sinh(s'z),$$

by using the method of integration given by Lord Rayleigh in his paper (*loc. cit. supra*), where we have put

$$h'^2 = \rho'p^2/(\lambda'+2\mu'),\quad \kappa'^2 = \rho'p^2/\mu',$$

and $$h'^2+r'^2 = l^2 = \kappa'^2+s'^2.$$

Also, since $$\Delta = \frac{\partial u}{\partial x}+\frac{\partial w}{\partial z},$$

we have $$ilX_1+s'Z_2 = 0,\quad ilX_2+s'Z_1 = 0.$$

In virtue of the rigid connexion between the two solids, we have, at $z = 0$, $$u = u' \text{ and } w = w',$$

i.e., $$-ilB+X_1 = -ilA+X,$$

$$-r'C+Z_1 = -rA+Z,$$

Also we have dynamical surface-conditions at $z=0$,

$$\lambda\Delta+2\mu\frac{\partial w}{\partial z}=\lambda'\Delta'+2\mu'\frac{\partial w'}{\partial z},$$

$$\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=\mu'\left(\frac{\partial u'}{\partial z}+\frac{\partial w'}{\partial x}\right),$$

and those at $z=h_0$, the free surface, are

$$\lambda'\Delta'+2\mu'\frac{\partial w'}{\partial z}=0,$$

$$\frac{\partial u'}{\partial z}+\frac{\partial w'}{\partial x}=0.$$

The conditions at $z=0$ yield

$$-(2l^2-\kappa^2)A+2sZ=(\mu'/\mu)\left[-(2l^2-\kappa'^2)B+2s'Z_2\right],$$

$$2l^2rA-(2l^2-\kappa^2)Z=(\mu'/\mu)\left[2l^2r'C-(2l^2-\kappa'^2)Z_1\right],$$

in which some reductions have been made by substituting for the X's their values in terms of the Z's.

The conditions at $z=h_0$ give, after expanding and retaining only the first powers of $r'h_0$, $s'h_0$,

$$-(2l^2-\kappa'^2)(B+Cr'h_0)+2s'(Z_2+Z_1s'h_0)=0,$$

$$2l^2r'(C+Br'h_0)-(2l^2-\kappa'^2)(Z_1+Z_2s'h_0)=0,$$

and here again we have substituted for the X's in terms of the Z's.

Thus we have

$$-(2l^2-\kappa'^2)B+2s'Z_2=h_0\left[(2l^2-\kappa'^2)Cr'-2s'^2Z_1\right],$$

$$2l^2r'C-(2l^2-\kappa'^2)Z_1=h_0\left[(2l^2-\kappa'^2)Z_2s'-2l^2r'^2B\right],$$

and, substituting these values in the dynamical conditions at $z=0$, we get

$$-(2l^2-\kappa^2)A+2sZ=(\mu'/\mu)h_0\left[(2l^2-\kappa'^2)Cr'-2s'^2Z_1\right],$$

$$2l^2rA-(2l^2-\kappa^2)Z=(\mu'/\mu)h_0\left[(2l^2-\kappa'^2)s'Z_2-2l^2r'^2B\right].$$

Now it must be observed that we have already rejected squares of h_0, and consequently it will be sufficiently accurate, when reducing the right in the last pair of equations, to entirely reject h_0 in the expressions for Z_1, Z_2.

Thus we take

$$(2l^2-\kappa'^2)\,Cr' - 2s'^2 Z_1 = \left[(2l^2-\kappa'^2)^2 - 4l^2 s'^2\right] Cr'/(2l^2-\kappa'^2)$$
$$= Cr'\kappa'^4/(2l^2-\kappa'^2),$$
$$(2l^2-\kappa'^2)\,Z_2 s' - 2l^2 r'^2 B = \left[(2l^2-\kappa'^2)^2 - 4l^2 r'^2\right] B/2$$
$$= B\left[\kappa'^4 - 4l^2(\kappa'^2-h'^2)\right]/2;$$

so that $$-(2l^2-\kappa^2)A + 2sZ = (\mu'/\mu)\,h_0 Cr'\kappa'^4/(2l^2-\kappa'^2),$$
$$2l^2 rA - (2l^2-\kappa^2)\,Z = (\mu'/\mu)\,h_0 B\left[\kappa'^4 - 4l^2(\kappa'^2-h'^2)\right]/2.$$

Next we must express C, B in terms of A, Z, and in doing so it will not be necessary to retain h_0, by the argument given before.

Now we have $$l^2B - s'Z_2 = l^2A - sZ,$$
$$r'C - Z_1 = rA - Z,$$
and, rejecting h_0, $$2s'Z_2 = (2l^2-\kappa'^2)\,B,$$
$$(2l^2-\kappa'^2)\,Z_1 = 2l^2 r'C;$$
thus we find $$\kappa'^2 B = 2\,(l^2A - sZ),$$
$$\kappa'^2 r'C = -(2l^2-\kappa'^2)(rA-Z).$$

As a last reduction I now eliminate Z from these values of B, C by substituting in terms of A, still neglecting h_0. Thence

$$\kappa'^2 B = \kappa^2 A,$$
and $$\kappa'^2 r'C/(2l^2-\kappa'^2) = \kappa^2 rA/(2l^2-\kappa^2).$$

Thus our equations connecting A, Z will become

$$-(2l^2-\kappa^2)A + 2sZ = (\mu'/\mu)\,h_0\kappa^2\kappa'^2 rA/(2l^2-\kappa^2),$$
$$2l^2 rA - (2l^2-\kappa^2)\,Z = (\mu'/\mu)\,h_0\kappa^2 A\left[\kappa'^4 - 4l^2(\kappa'^2-h'^2)\right]/2\kappa'^2.$$

Now, eliminating the ratio $A : Z$, we have

$$4l^2 rs - (2l^2-\kappa^2)^2 = (\mu'/\mu)\,h_0\left[\kappa^2\kappa'^2(r+s) - 4l^2\kappa^2 s\,(1-h'^2/\kappa'^2)\right].$$

Writing, as before, $$\kappa^2/l^2 = \zeta,$$
and $$h^2/\kappa^2 = \tau, \quad h'^2/\kappa'^2 = \tau',$$
this becomes

$$4\left[(1-\tau\zeta)(1-\zeta)\right]^{\frac{1}{2}} - (2-\zeta)^2$$
$$= lh_0\zeta\left[(\rho'/\rho)\,\zeta\left\{(1-\zeta)^{\frac{1}{2}} + (1-\tau\zeta)^{\frac{1}{2}}\right\} - 4(\mu'/\mu)(1-\tau')(1-\zeta)^{\frac{1}{2}}\right];$$

and, to reduce this further, we may insert on the right values found

by equating the left to zero; and it will be found that, if

$$(2-\zeta_0)^2 = 4(1-\zeta_0)^{\frac{1}{2}}(1-\tau\zeta_0)^{\frac{1}{2}},$$

then $$\zeta_0\left[(1-\zeta_0)^{\frac{1}{2}}+(1-\tau\zeta_0)^{\frac{1}{2}}\right] = 4(1-\tau)(1-\zeta_0)^{\frac{1}{2}}.$$

Thus $4(1-\tau\zeta)^{\frac{1}{2}}(1-\zeta)^{\frac{1}{2}}-(2-\zeta)^2$

$$= 4lh_0\zeta_0(1-\zeta_0)^{\frac{1}{2}}\left[(\rho'/\rho)(1-\tau)-(\mu'/\mu)(1-\tau')\right]$$

is the new form of our equation.

To solve approximately, write

$$\zeta = \zeta_0+\delta\zeta,$$

and then we have

$$-(\delta\zeta/\zeta_0)\left[(1+\tau-2\tau\zeta_0)(1-\tau\zeta_0)^{-\frac{1}{2}}(1-\zeta_0)^{-\frac{1}{2}}-(2-\zeta_0)\right]$$
$$= 2lh_0(1-\zeta_0)^{\frac{1}{2}}\left[(\rho'/\rho)(1-\tau)-(\mu'/\mu)(1-\tau')\right].$$

If, now, V_0 be the velocity of propagation of these waves in the elastic solid when free from the shell, and $V_0+\delta V$ be the velocity of propagation now found, we have

$$V_0^2 = p_0^2/l^2 = \mu\kappa_0^2/\rho l^2 = \zeta_0\mu/\rho,$$

and $$(V_0+\delta V)^2 = (\zeta_0+\delta\zeta)\,\mu/\rho;$$

thus $$2\delta V/V_0 = \delta\zeta/\zeta_0$$

approximately.

Lord Rayleigh has given the appropriate roots of

$$4(1-\tau\zeta)^{\frac{1}{2}}(1-\zeta)^{\frac{1}{2}} = (2-\zeta)^2$$

for four values of τ; and, using these values of ζ_0, I have found roughly

$$\delta V/V_0 = (0{\cdot}13)\,lh_0\left[(\mu'/\mu)(1-\tau')-\rho'/\rho\right], \quad \tau = 0,$$
$$\delta V/V_0 = (0{\cdot}34)\,lh_0\left[(\mu'/\mu)(1-\tau')-2\rho'/3\rho\right], \quad \tau = 1/3,$$
$$\delta V/V_0 = (0{\cdot}70)\,lh_0\left[(\mu'/\mu)(1-\tau')-\rho'/2\rho\right], \quad \tau = 1/2,$$
$$\delta V/V_0 = (2{\cdot}80)\,lh_0\left[(\mu'/\mu)(1-\tau')-\rho'/4\rho\right], \quad \tau = 3/4.$$

It thus appears that the influence of a thin skin on the velocity of propagation of waves of given wave-length can be only slight; hence any application of Lord Rayleigh's results to determine the velocities of earthquake waves cannot be expected to agree at all closely with the values observed until we know something of the elastic constants of the earth at depths comparable with the wave-length.

Some Multiform Solutions of the Partial Differential Equations of Physical Mathematics and their Applications. By H. S. Carslaw.

Received and read November 10th, 1898. Received, in revised form, January 20th, 1899.

CONTENTS.

Introduction.

This paper owes its origin to my work in the University of Göttingen in the Summer Semester of 1897. The study of a memoir by Professor Sommerfeld, then a *Privat-docent* in that University, suggested to me the possibility, by a somewhat similar method, of obtaining multiform solutions of other differential equations of physical mathematics. Their applications are not far to seek. In conversation with Dr. Sommerfeld on the subject, he told me that this field for research had been pointed out by him at the close of his paper communicated on April 10th of that year to this Society, and then in the press. However, as his time was fully occupied with other work, he most generously urged me to take up

the investigation, and offered me his help if at any time the obscurities of the subject left me in difficulty. I desire at the outset to express the sense of my gratitude for this great kindness, and for the readiness with which he removed some of the difficulties which faced me at the beginning of my work.

The papers to which I have referred, and to which fuller reference will be made immediately, contain certain multiform solutions of the equations

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\kappa^2 u=0$$

and

$$\nabla^2 u=0.$$

The solutions of the first are applied to the two-dimensional problem of the Diffraction and Reflection of Plane Waves of Light incident on an opaque semi-infinite plane bounded by a straight edge. Of this problem Lord Rayleigh had stated some years before, in the article on "Wave Theory" in the *Encyclopædia Britannica*, that its mathematical difficulties were so formidable that no successful attempt had yet been made to solve it; while again, in his *Theory of Sound*,* he has called attention to the claims of such questions involving diffraction.

The solutions of the second equation find their application in such electrical or hydrodynamical problems as deal with this boundary.

The advance made, in this paper, is the determination of corresponding multiform solutions for the equations

$$\nabla^2 u+\kappa^2 u=0$$

and

$$\frac{\partial u}{\partial t}=\kappa\nabla^2 u,$$

and their application to problems in the theories of sound and conduction of heat. The solutions obtained are exact, and expressed as definite integrals. The work is thus on a different plan from the most important memoirs of Poincaré, "Sur la Polarisation par Diffraction,"† and Lamb, on "The Reflection and Transmission of Electric Waves by a Metallic Grating,"‡ in both of which the results are obtained in series and by approximation.

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* *Theory of Sound*, Vol. II., p. 141, 2nd ed.
† *Acta Mathematica*, Bd. XVI., p. 297; Bd. XX., p. 313.
‡ *Proc. Lond. Math. Soc.*, Vol. XXIX., p. 523.

1. *Extension of the Method of Images.*

The method of images, taken from the domain of optics and applied to the solution of certain problems in statical electricity, was soon extended into other branches of applied mathematics. Instances of its application occur in current electricity, hydrodynamics, and the theory of the conduction of heat. The principle of the method is the symmetrical extension* of the problem involved, from the limited to the unlimited space. Thus the question of the point charge between two planes at right angles is solved by the consideration of the infinite space, and charges at the four symmetrical points. This symmetrical extension is obtained by successively reflecting the original space in the bounding planes. By this means the whole space is simply and completely filled up, while the starting point is reproduced in the final reflection. Similarly with the space between two infinite planes meeting at an angle $\frac{1}{3}\pi$. Here six reflections are required before we return to the region from which we started. Fig. 1 shows the position of the poles for this

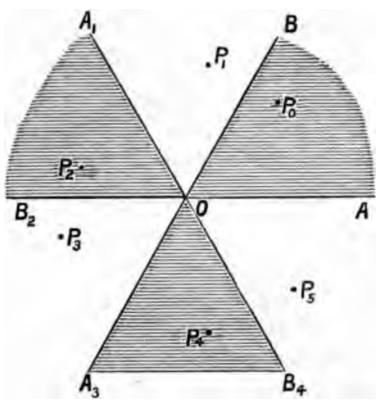

FIG. 1.—Position of charge and images for planes inclined at an angle $\frac{1}{3}\pi$, obtained by successive reflection.

case, the shaded portions being those in which the positive charges are placed.

The result for the angle $\frac{\pi}{m}$ (m a positive integer) follows in the same way.

When we attempt, by this method, to solve the problems in which the angle between the planes is $\frac{n\pi}{m}$ (n, m positive integers), we

* *Analytische Fortsetzung.*

at once meet a difficulty; on reproducing the original space by successive reflection, we have, in the end, more than one pole in the region from which we started. In other words, *the space is not simply filled up, but we are compelled to traverse it n times before we return to our starting point.*

For $\frac{2}{3}\pi$ (Fig. 2) our space is covered two-fold, and we have six reflections. These six are all necessary, as, though the second brings us the complete revolution, the third does not take the starting point back to its original position. The spaces are here shaded, or otherwise, according as the positive or negative charges occur, and we find ourselves with two poles in the region which ought only to possess one.

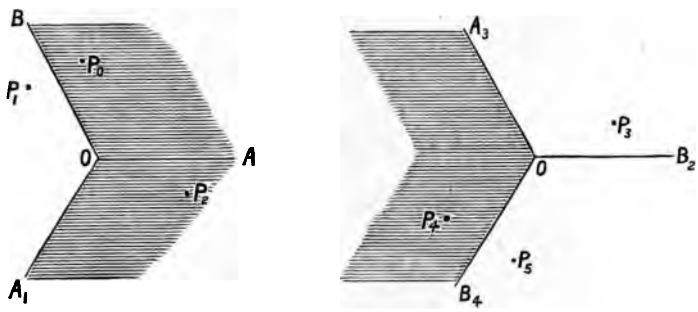

FIG. 2.—Planes inclined at an angle $\frac{2}{3}\pi$.

The method of images, then, seems here to fail.

The first successful attempt to solve any of these problems in mathematical physics appears to have been made in 1894 by Sommerfeld. This was published in a paper, "On the Analytical Theory of the Conduction of Heat,"* *Mathematische Annalen*, Bd. XLV. The ideas there introduced were extended to optics and electricity in a paper in the same journal, Bd. XLVII., "On the Mathematical Theory of Diffraction."† Some of the results of this paper had already been communicated to the Königl. Gesellschaft der Wissenschaften zu Göttingen, and appear in its *Nachrichten* in the communications noted below.‡ The method is somewhat altered, and

* "Zur analytischen Theorie der Wärme-leitung," *Math. Ann.*, Bd. XLV.

† "Mathematische Theorie der Diffraction," *Math. Ann.*, Bd. XLVII. A review of this paper will be found in Voigt's *Kompendinm der theor. Physik*, Bd. II., pp. 766–776.

‡ "Zur mathematischen Theorie der Beugungserscheinungen," *Nachrichten von der Königl.-Gesellschaft der Wissenschaften*, Göttingen, 1894. "Zur Integration der partiellen Differential-Gleichung $\nabla^2 u + k^2 u = 0$ auf Riemanns'chen Flächen," ditto, 1895.

brought to bear on potential problems, in a paper "On Multiform Potential in Space"* communicated to this Society.

As used in this last paper, the method may be briefly stated thus. We imagine that we are dealing not with the ordinary space but with a Riemann's space. This is analogous to the Riemann's surface of the theory of functions of a complex variable, and allows us to look upon such many-valued functions in the ordinary space as single-valued in the Riemann's space. In space we shall have "branch-lines"† instead of "branch-points"‡; "branch-membranes"§ for "branch-sections."|| Every plane section of the Riemann's space will give a Riemann's surface, and the branch-membranes and branch-lines give place to branch-sections and branch-points. We then attempt to find a multiform solution of the differential equation —in this case $\nabla^2 u = 0$—which shall be uniform in the Riemann's space; in other words, our problem, from the pure mathematical point of view, is simply the integration of this partial differential equation in a suitable Riemann's space. Finally, we obtain a function u which has the following properties:—

(i.) *In the Riemann's space outside the branch-lines it is single-valued, finite, and continuous, except in the point P, where it is infinite as $\frac{1}{R}$, R denoting the distance from P to the neighbouring point Q.*

(ii.) *It satisfies the differential equation $\nabla^2 u = 0$ in the whole Riemann's space except in P, and in the branch-lines. In this condition is included the fact that, except in these places, it has finite first and second differential coefficients.*

(iii.) *It vanishes at infinity.*

By taking the images, and considering the space we have to deal with as the Riemann's space, we obtain a potential function with n poles; but, taking the physical space as that given by but one "example"¶ of the Riemann's space, we have the solution of our problem.

For example, take the case solved by Sommerfeld, of the point charge outside a semi-infinite conducting plane at zero potential.

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* "Über verzweigte Potentiale im Raum," *Proc. Lond. Math. Soc.*, Vol. XXVIII.

† *Verzweigungslinien.* ‡ *Verzweigungspunkte.*

§ *Verzweigungsmembranen.* || *Verzweigungsschnitte.*

¶ *Exemplar.*

Here the convenient Riemann's space has the edge of the plane—the axis of z—for branch-line, and the plane itself for branch-membrane. Then, with cylindrical coordinates, we take the range

$$0 < \theta < 2\pi$$

for the physical space; and

$$-2\pi < \theta < 0$$

for the imaginary space, the two building up the twofold Riemann's space.

A solution is found, corresponding to the pole at (r', θ', z'),

$$0 < \theta' < 2\pi,$$

and it is proved that there is only one solution with these properties.

Denoting this by $u(\theta')$,

$$\bar{u} = u(\theta') - u(-\theta')$$

is the required solution of the physical problem.

This paper contains some further extensions of this method.

From the pure mathematical point of view, it deals with the solution on certain Riemann's surfaces, and, in corresponding Riemann's spaces, of the following partial differential equations:—

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \kappa^2 u = 0,$$

$$\nabla^2 u + \kappa^2 u = 0,$$

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u.$$

From the physical standpoint, it is concerned with problems in which the ordinary image theory fails, and the space concerned has to be looked upon as a Riemann's space (or surface), of which only one example (or sheet) is considered.

2. *Multiform Solution of the Equation* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \kappa^2 u = 0$, *without Infinity.*

The solution discussed in this section forms the subject of the paper on "Diffraction," in *Math. Ann.*, Bd. XLVII., above cited. The results are so important—they solve the problem of the diffraction of electrical waves incident on a semi-infinite plane conducting screen—that it seems worth while to obtain the solution anew, and, in

obtaining it, more fully to explain the method hereafter to be employed. Whereas, in these companion papers in *Math. Ann.* and *Gött. Nachrichten*, the solutions of the two-dimensional case are obtained as limiting results from three-dimensional work, just as Bessel's Functions can be deduced from Spherical Harmonics, it is obvious, from the paper on "Potential,"* that nothing hinders the application of its method to the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\kappa^2 u=0.$$

A comparison of the work in this section with that in the paper referred to will show to what an extent the problem is simplified.

In dealing with plane waves we are accustomed to the solution

$$u_0=e^{ikr\cos(\theta-\theta')}, \tag{1}$$

which represents the disturbance due to waves coming in the direction (θ') from infinity.

If we introduce the complex variable α, and let $f(\alpha)$ stand for any function of α,

$$\int e^{ikr\cos(\alpha-\theta)} f(\alpha)\, d\alpha,$$

taken over any path in the α-plane from which infinities are excluded, is also a solution.

Then
$$\frac{1}{2\pi}\int e^{ikr\cos(\alpha-\theta)}\frac{e^{i\alpha}}{e^{i\alpha}-e^{i\theta'}}\,d\alpha, \tag{2}$$

taken over any circuit in the α-plane, surrounding the point $\alpha=\theta'$ and no other singularity of the integrand, is, by Cauchy's Theorem, the same as u_0; and we have an identical transformation. We may deform this path—provided that in doing so we do not pass over any of the singular points of the integrand.

There is no trouble here about branch-points† because the function to be integrated is uniform.

Since $\cos(\alpha-\theta)=\cos(a-\theta)\cosh b-i\sin(a-\theta)\sinh b,$

when $\alpha=a+ib$, we see that we may deform the path to infinity along the imaginary axis, provided that

for $b=+\infty$, $\sin(a-\theta)$ be negative,

and for $b=-\infty$, $\sin(a-\theta)$ be positive ;

* *Cf. Proc. Lond. Math. Soc.*, Vol. XXVIII., p. 429.

† *Verzweigungspunkte.*

for the real part of the exponential is $e^{\kappa r \sin(a-\theta)\sinh b}$, and when $b = +\infty$, $\sinh b = +\infty$, while, when $b = -\infty$, $\sinh b = -\infty$.

Now we may consider, in the first instance, that in the physical space $|\theta - \theta'| < \pi$. This only compels us to make our current co-ordinate θ lie within the range $-(\pi - \theta') < \theta < (\pi + \theta')$.

In Fig. 4 the shaded portions represent the parts of the a-plane where our path may reach infinity. The curve drawn is a possible deformation of the original circuit round $a = \theta'$.

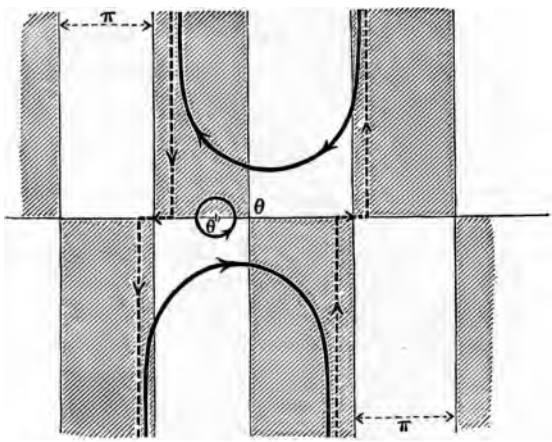

FIG. 3.—Breadth of strip, π; deformation of circuit round $a = \theta'$; $|\theta - \theta'| < \pi$; $n = 1$.

The breadth of the strips is π. The parts of the path made up of straight lines, dotted in the figure, are separated by 2π. This enables us to leave these out of account, owing to the periodicity by 2π of the integrand, and the fact that the corresponding parts are described in opposite directions. The curved parts are to be asymptotic to these lines. It will be easily seen that any other path, starting to the left of the circuit round θ' and ending at distance 2π on the right, will be deformable into this.

These two *curved branches* we call, after Sommerfeld, the path (A) corresponding to the value of θ.

We have here proved that

$$\frac{1}{2\pi}\int e^{ikr\cos(a-\theta)}\frac{e^{ia}}{e^{ia}-e^{i\theta'}}\,da,$$

* *Verzweigungspunkte.*

over the path (A), is equal to

$$e^{i\kappa r\cos(\theta-\theta')},$$

and this solution is uniform.

We now proceed to the Multiform Solution.

Consider the function defined by

$$u = \frac{1}{2\pi n}\int e^{ikr\cos(\alpha-\theta)} \frac{e^{i\alpha/n}}{e^{i\alpha/n}-e^{i\theta'/n}}\, d\alpha, \tag{3}$$

the integral being taken over the path (A), in the α-plane, which corresponds to the value of the current coordinate θ.

(i.) *This function is a solution of our equation*, since every element of the integrand is a solution, and we have excluded the possibility of infinite values. Also, when $n = 1$, it takes the form

$$u = u_0 = e^{ikr\cos(\theta-\theta')}.$$

(ii.) *The function is multiform, and of period* $2n\pi$, in the ordinary sense; but on the n-sheeted Riemann's surface with the origin as branch-point, and the line $\theta = -(\pi-\theta')$ as branch-section, it is uniform.

To prove this we must again have recourse to Fig. 3.

When we put for θ, $\theta+2\pi$, the alteration on the path (A) is simply to move it parallel to the axis of imaginary quantities through a distance 2π. Thus a change in θ of $2n\pi$, or n revolutions round the axis of z, moves the path (A) along the real axis of α through $2n\pi$.

Now the integrand is periodic in θ and of period $2n\pi$; therefore the values assigned at each point of the path for $\theta+2n\pi$ are the same as those at corresponding points for θ. Thus the value of u for the point (r, θ) is the same as for the point $(r, \theta+2n\pi)$.

(iii.) *It is finite and continuous for all real finite values of r.*

That the function is continuous follows from the fact that a slight change in θ only displaces through an infinitesimal amount the path of the integration, and only alters the integrand infinitesimally. That it is finite follows from the way in which we have chosen the path.

(iv.) *Further, at infinity in the first sheet, i.e.* ($r = \infty$, $|\theta-\theta'| < \pi$), $u = u_0$, *and, in the other sheets*, $u = 0$.

In speaking of the different sheets of the Riemann's surface, we only mean that at each complete revolution on passing over $\theta = -(\pi-\theta')$, or $\pi+\theta$, $3\pi+\theta'$, &c., we are passing from one sheet to the other.

To prove the proposition it is sufficient to note that the paths (A), corresponding to points on the second, third, &c., sheets, may be deformed to the rectilinear portions alone, as no pole of the integrand lies in the portion of the a-plane enclosed. These portions lie wholly in the shaded parts of the plane, and therefore, when $r = \infty$, vanish. On the other hand, for points at infinity on the first sheet, $u = u_0$, since, in addition to the rectilinear portion, our path (A) gives a circuit round the pole $a = \theta'$. This is plain from Fig. 4.

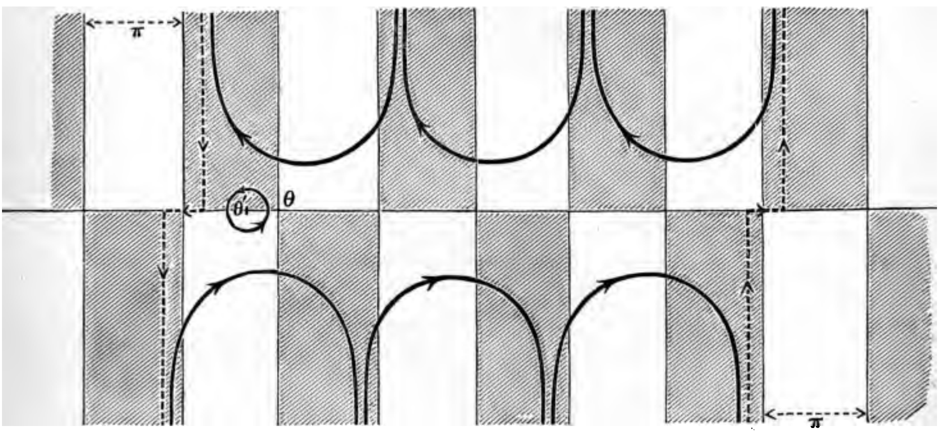

FIG. 4.—Breadth of strip, π; deformation of circuit round $a = \theta'$;
$|\theta - \theta'| < \pi$; $n = 3$.

(v.) *If* $u_1, u_2, u_3, \ldots, u_n$ *be the values of* u *at underlying points on the Riemann's surface—in other words, at the points* (r, θ), $(r, \theta + 2\pi)$, *&c.*—

$$u_1 + u_2 + \ldots + u_n = u_0.$$

To prove this we have only to give the accompanying figure containing the paths corresponding to $u_1, u_2, \ldots, u_n$ for $n = 3$. These paths may be joined at $b = \pm\infty$, and we may introduce the rectilinear portions separated by $2n\pi$ (*i.e.*, 6π) without altering the sum $u_1 + u_2 + \ldots + u_n$. The integral over the completed path, by Cauchy's theorem, is the same as u_0, the only pole enclosed being at $a = \theta'$.

To sum up, we have found a function u which has the following properties:—

(i.) *It is a solution of our differential equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \kappa^2 u = 0.$$

(ii.) *It is uniform on the n-sheeted Riemann's surface; or, in other words, periodic of period $2n\pi$ in θ.*

(iii.) *It is finite and continuous for all real values of r.*

(iv.) *It is equal to u_0 at infinity on the first sheet, i.e., when $|\theta-\theta'|<\pi$ and $r=\infty$, $u=e^{i\kappa r\cos(\theta-\theta')}$; on the other sheets it is zero at infinity, i.e., when $\pi<|\theta-\theta'|<3\pi$, $3\pi<|\theta-\theta'|<5\pi$, ..., $(2n-3)\pi<|\theta-\theta'|<(2n-1)\pi$, and $r=\infty$, $u=0$.*

(v.) *The n values at the corresponding points on the n sheets satisfy the condition $u_1+u_2+\ldots+u_n=u_0$.*

Calculation of the Value of u for $n=2$.

It would be possible to calculate the value of u for a point on any one of the sheets and for any value of n. However, the chief interest of the problem lies in the case $n=2$.

Consider any value of θ', and suppose that we wish to find the values of u at underlying points on the Riemann's surface. We thus allow θ to move from $-(\pi-\theta')$ to $(3\pi+\theta')$. On the first sheet

$$-(\pi-\theta')<\theta<(\pi+\theta');$$

on the second $$(\pi+\theta')<\theta<(3\pi+\theta').$$

Let the values of u at corresponding points be denoted by u_1 and u_2.

Then $$u_1+u_2=u_0.$$

Also, u_2 is easily evaluated. We replace the two curved portions of the path (A) by the rectilinear parts, and these in turn by the lines $a=\theta+\pi$ and $\theta+3\pi$, taken in opposite directions. Thus

$$u_2=\frac{1}{4\pi}\int_{-\infty}^{\infty}e^{-i\kappa r\cosh b}\left(\frac{1}{1-e^{\frac{1}{2}[i(\theta'-\theta-ib-\pi)]}}-\frac{1}{1-e^{\frac{1}{2}[i(\theta'-\theta-ib-3\pi)]}}\right)idb$$

$$=\frac{1}{4\pi}\int_{-\infty}^{\infty}e^{-i\kappa r\cosh b}\left(\frac{1}{1+ie^{\frac{1}{2}[i(\theta'-\theta-ib)]}}-\frac{1}{1-ie^{\frac{1}{2}[i(\theta'-\theta-ib)]}}\right)idb$$

$$=\frac{1}{4\pi}\int_{-\infty}^{\infty}e^{-i\kappa r\cosh b}\frac{1}{\cos\frac{1}{2}(\theta'-\theta-ib)}\,db$$

$$=\frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta')\int_{0}^{\infty}e^{-i\kappa r\cosh b}\frac{\cosh\frac{1}{2}b}{\cosh b+\cos(\theta-\theta')}\,db; \tag{4}$$

therefore

$$\frac{u_2}{u_0} = \frac{1}{\pi} \cos \tfrac{1}{2} (\theta - \theta') \int_0^\infty e^{-i\kappa r [\cosh b + \cos (\theta - \theta')]} \frac{\cosh \frac{1}{2} b}{\cos b + \cos (\theta - \theta')} \, db$$

$$= X,^{*} \text{ say};$$

therefore

$$\frac{\partial X}{\partial r} = -\frac{i\kappa}{\pi} \cos \tfrac{1}{2} (\theta - \theta') \, e^{-2i\kappa r \cos^2 \frac{1}{2} (\theta - \theta')} \int_0^\infty e^{-2i\kappa r \sinh^2 \frac{1}{2} b} \cosh \tfrac{1}{2} b \, db$$

$$= -\frac{i\kappa}{\pi} \cos \tfrac{1}{2} (\theta - \theta') \, e^{-2i\kappa r \cos^2 \frac{1}{2} (\theta - \theta')} \sqrt{\frac{\pi}{2i\kappa r}};$$

therefore

$$\frac{\partial X}{\partial r} = -\frac{1}{\sqrt{\pi}} e^{\frac{1}{4} i\pi} \frac{\partial}{\partial r} \left[\int_0^{\sqrt{2\kappa r} \cos \frac{1}{2} (\theta - \theta')} e^{-i\lambda^2} \, d\lambda \right];$$

therefore

$$X = -\frac{1}{\sqrt{\pi}} e^{\frac{1}{4} i\pi} \int_0^{\sqrt{2\kappa r} \cos \frac{1}{2} (\theta - \theta')} e^{-i\lambda^2} \, d\lambda + X_0, \tag{5}$$

where X_0 is the value of X for $r = 0$.

This is easily found to be $\frac{1}{2}$, *i.e.*,

$$\frac{e^{\frac{1}{4} i\pi}}{\sqrt{\pi}} \int_0^\infty e^{-i\lambda^2} \, d\lambda, \quad \text{or} \quad \frac{e^{\frac{1}{4} i\pi}}{\sqrt{\pi}} \int_{-\infty}^0 e^{-i\lambda^2} \, d\lambda.$$

Hence

$$u_2 = u_0 \frac{e^{\frac{1}{4} i\pi}}{\sqrt{\pi}} \int_{-\infty}^{-T} e^{-i\lambda^2} \, d\lambda, \tag{6}$$

and

$$u_1 = u_0 \frac{e^{\frac{1}{4} i\pi}}{\sqrt{\pi}} \int_{-\infty}^{+T} e^{-i\lambda^2} \, d\lambda, \tag{7}$$

where

$$T = \sqrt{2\kappa r} \cos \tfrac{1}{2} (\theta - \theta').$$

It is to be noticed that the value u_2 is that at the point $(r, \theta + 2\pi)$ in the second sheet, since u_1 is found for the point (r, θ). Reducing this to the current coördinates, we have on the second sheet, at (r, θ),

$$u = u_0 \frac{e^{\frac{1}{4} i\pi}}{\sqrt{\pi}} \int_{-\infty}^{+T} e^{-i\lambda^2} \, d\lambda,$$

the same form as for u at the point (r, θ) on the first sheet.

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* *Cf. Math. Ann.*, Bd. XLVII., p. 358.

Thus we have found that

$$u = \frac{e^{i[\kappa r\cos(\theta-\theta')+\frac{1}{4}\pi]}}{\sqrt{\pi}} \int_{-\infty}^{T} e^{-i\lambda^2}\, d\lambda, \tag{8}$$

where $$T = \sqrt{2\kappa r} \cos \tfrac{1}{2}(\theta-\theta'),$$

is a finite and continuous solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \kappa^2 u = 0,$$

which is periodic in θ and of period 4π; and that at $r = \infty$, when $|\theta-\theta'| < \pi$, it takes the form $e^{i\kappa r\cos(\theta-\theta')}$, while, when

$$\pi < |\theta - \theta'| < 3\pi,$$

it is zero.

Since our solutions are reduced to the same form and are of period 4π, we are able to remove the condition that θ lies between $-(\pi-\theta')$ and $(3\pi+\theta')$, and take the more convenient range from -2π to $+2\pi$. In considering the value at infinity of the function we shall still need to note in which sheet of the surface the point lies; in other words, whether, for the required values of θ and θ', $\cos\frac{1}{2}(\theta-\theta')$ is positive or negative.

This is the solution found by another method by Sommerfeld, in his paper on "Diffraction."

3. *Application to the Theory of Sound.—The Problem of the Diffraction of Plane Waves of Sound incident on a Thin Semi-infinite Rigid Plane bounded by a Straight Edge.*

Taking ϕ for the velocity potential of the medium in which the velocity of sound is V, we know that it satisfies the equation

$$\frac{\partial^2 \phi}{\partial t^2} = V^2 \nabla^2 \phi. \tag{9}$$

To solve this in the case of periodic motion we may assume

$$\phi = \text{real part of } (u \,.\, e^{2i(\pi/\tau)t}), \tag{10}$$

and we find for u the equation

$$\nabla^2 u + \kappa^2 u = 0,$$

where $$\kappa^2 = \frac{4\pi^2}{\tau^2 V^2}.$$

Thus our equation for two-dimensional motion takes the form of that

of last section. The solution found is applicable to the case in which we have plane waves of sound coming from the direction $\theta = \theta'$, and incident on the plane, which we take as $\theta = 0$ $(0 < r < \infty)$.

This problem is fully discussed in Sommerfeld's paper.* The waves there are supposed to be electro-magnetic or optical. The solution is obtained by adding† the multiform solutions of period 4π for waves from the directions (θ') and $(-\theta')$; *i.e.*,

$$\bar{u} = \frac{e^{i(\frac{1}{4}\pi)}}{\sqrt{\pi}} \left(e^{i\kappa r \cos(\theta-\theta')} \int_{-\infty}^{\sqrt{2\kappa r}\cos\frac{1}{2}(\theta-\theta')} e^{-i\lambda^2}\, d\lambda + e^{i\kappa r\cos(\theta+\theta')} \int_{-\infty}^{\sqrt{2\kappa r}\cos\frac{1}{2}(\theta+\theta')} e^{-i\lambda^2}\, d\lambda \right), \tag{11}$$

where the physical space is taken as given by

$$0 < \theta < 2\pi,$$

and within it $\bar{u}$ satisfies all the conditions.

Sommerfeld finds approximations for the results, when r is great. He proves that the space has to be considered in five sections: namely,

(i.) That from $\theta = 0$ to a parabola with the line $(\pi - \theta')$ as axis, the pole for focus, and extremely small parameter;

(ii.) The area enclosed by this parabola;

(iii.) The area between this parabola and a similar one at $\pi + \theta'$;

(iv.) The area enclosed by this curve; and, lastly,

(v.) That between this curve and $\theta = 2\pi$.

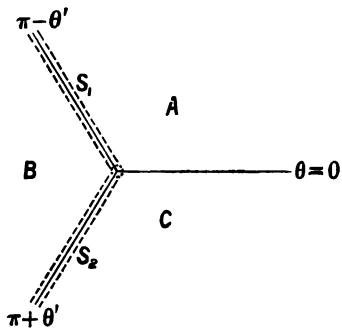

Fig. 5.

* *Math. Ann.*, Bd. XLVII., pp. 368, 369.

† In the sound problem we consider only the case in which the two solutions are added. In the optical there is also a physical interpretation of the results obtained by subtraction.

He finds, when r is very great, that in these divisions A, B, C the following approximations hold:

$$\text{(A)}\quad \phi = \cos 2\pi\left(\frac{r}{\lambda}\cos(\theta-\theta')+\frac{t}{\tau}\right)+\cos 2\pi\left(\frac{r}{\lambda}\cos(\theta+\theta')+\frac{t}{\tau}\right)$$

$$-\frac{1}{4\pi}\left\{\cos\left[2\pi\left(\frac{r}{\lambda}-\frac{t}{\tau}\right)+\frac{\pi}{4}\right]\sqrt{\frac{\lambda}{r}}\left(\frac{1}{\cos\frac{1}{2}(\theta+\theta')}+\frac{1}{\cos\frac{1}{2}(\theta-\theta')}\right)\right\}. \tag{12}$$

$$\text{(B)}\quad \phi = \cos 2\pi\left(\frac{r}{\lambda}\cos(\theta-\theta')+\frac{t}{\tau}\right)$$

$$-\frac{1}{4\pi}\left\{\cos\left[2\pi\left(\frac{r}{\lambda}-\frac{t}{\tau}\right)+\frac{\pi}{4}\right]\sqrt{\frac{\lambda}{r}}\left(\frac{1}{\cos\frac{1}{2}(\theta+\theta')}+\frac{1}{\cos\frac{1}{2}(\theta-\theta')}\right)\right\}. \tag{13}$$

$$\text{(C)}\quad \phi = -\frac{1}{4\pi}\left\{\cos\left[2\pi\left(\frac{r}{\lambda}-\frac{t}{\tau}\right)+\frac{\pi}{4}\right]\sqrt{\frac{\lambda}{r}}\left(\frac{1}{\cos\frac{1}{2}(\theta+\theta')}+\frac{1}{\cos\frac{1}{2}(\theta-\theta')}\right)\right\}. \tag{14}$$

In S_1 and S_2 we have to refer to the integrals.

These results throw light on the physical problem and illustrate the fact that the continued presence of the incident gives rise to reflected and diffracted waves.*

It is interesting to note that there is in the solution, as might be expected, infinite velocity at the sharp edge $r=0$. This is evident from the value of u in the integral form, and the velocity components will be found to contain $\dfrac{1}{\sqrt{r}}$.

4. *Multiform Solution of the Equation $\nabla^2 u+\kappa^2 u=0$, with an Infinity at a Point at a Finite Distance from the Origin.*

In the last two sections we have treated of a finite multiform solution of this equation in two dimensions which may be applied to the problem of plane waves incident on a thin rigid semi-infinite plane bounded by a straight edge. From the physical standpoint we ought now to examine the case of a source of sound, or a vibratory source of any kind, in two dimensions with the same obstacle. We should have the same differential equation to solve, and our solution would need to be of period 4π in θ, and finite and continuous for finite values of r, except at the point where the source is situated, where it

* See the remarks on these results, *Math. Ann*, Bd. XLVII., pp. 369, 370.

must be infinite as $\log x$, when $x=0$. However, from the pure mathematical point of view, the three-dimensional case is much the simpler. No introduction of Bessel's Functions of the Second Kind is necessary. We shall examine this now, and return to the two-dimensional later.

To speak analytically, we desire a solution of the equation $\nabla^2 u + \kappa^2 u = 0$, with the following properties:—

(i.) *In our n-fold Riemann's space with the axis of z as branch-line, and the plane $\theta = 0$ as branch-membrane, it is to be uniform; in other words, it is to be periodic in θ and of period $2n\pi$.*

(ii.) *It is to be infinite as $\dfrac{e^{-i\kappa R}}{R}$, when $R=0$, at the point (r', θ', z') in the first example, where R stands for the distance from (r', θ', z') to the neighbouring point.*

(iii.) *It is to be finite and continuous for all real finite values of r in all the examples, except at the above-mentioned point.*

(iv.) *It is to be zero at infinity.*

The method of obtaining such a solution is perfectly analogous to that employed in § 2, and in Sommerfeld's paper on "Potential." Starting from the solution

$$u_0 = \frac{e^{-ik\sqrt{r^2+r'^2+(z-z')^2-2rr'\cos(\theta-\theta')}}}{\sqrt{r^2+r'^2+(z-z')^2-2rr'\cos(\theta-\theta')}}, \tag{15}$$

we proceed to the integral

$$\frac{1}{2\pi}\int \frac{e^{-ik\sqrt{2rr'[\cosh\alpha_1-\cos(\alpha-\theta)]}}}{\sqrt{2rr'[\cosh\alpha_1-\cos(\alpha-\theta)]}}\,\frac{e^{i\alpha}}{e^{i\alpha}-e^{i\theta'}}\,d\alpha \tag{16}$$

taken round a circuit in the α-plane enclosing $\alpha = \theta'$, and no other singularity, or branch-point, of the integrand.

We have now to deal with branch-points, because the radical sign has brought a multiform function of α into our integrand; further, we have written $\cosh\alpha_1$ for $\dfrac{r^2+r'^2+(z-z')^2}{2rr'}$.

With the above restrictions, this integral, by Cauchy's Theorem, is the same as u_0.

We can deform the path of integration in the α-plane without affecting the value of the integral, provided that we do not deform it over any of the singular points or branch-points of the function integrated; in this condition is contained the restriction from deform-

ing our path to points where the function would be infinite. Also, since we are dealing with a multiform function of the complex variable α, we must fix the value to be assigned to the function—in other words, the sign of the root—at a particular point of the path, and see that the values we assign to it at all points of the deformed path are those belonging to the "branch" of the function we are following. If we make sure of these things, we may treat the integrand as single-valued, and apply to it Cauchy's Theorem and its extensions. This requires only the definiteness and continuity of the function to be integrated.

Since we are dealing primarily with the ordinary space, we may suppose $|\theta-\theta'| < \pi$, which means that, in the first instance, we think of θ as varying from $-(\pi-\theta')$ to $(\pi+\theta')$, a full range of 2π.

The singularities of the integrand are given by

$$\alpha = 2m\pi + \theta', \quad \alpha = 2m\pi + \theta \pm i\alpha_1$$

(m, any integer), and the latter are branch-points.

The simplest method of determining the continuity of the values of $\sqrt{2rr'[\cosh\alpha_1 - \cos(\alpha-\theta)]}$, which we shall denote by R, is obtained from the consideration of the conformal representation of the α-plane on the R-plane.

Starting with

$$R = +\sqrt{2rr'(\cosh\alpha_1 + \cosh\infty)} = +\infty, \quad \text{for } \alpha = \theta - \pi + i\infty,$$

we proceed through

$$R = +\sqrt{2rr'(\cosh\alpha_1 + \cosh b)}, \qquad \text{for } \alpha = \theta - \pi + ib,$$

$$R = +\sqrt{2rr'(\cosh\alpha_1 + 1)}, \qquad \text{for } \alpha = \theta - \pi,$$

$$R = +\sqrt{2rr'(\cosh\alpha_1 - 1)}, \qquad \text{for } \alpha = \theta,$$

$$R = +\sqrt{2rr'(\cosh\alpha_1 - \cosh b)}, \qquad \text{for } \alpha = \theta + ib \ (b < \alpha_1);$$

and, if we took $\alpha = \theta + i\alpha_1$, we should find $R = 0$.*

However, since $\alpha = \theta + i\alpha_1$ is a branch-point, we suppose that a small circuit is described from the point $\alpha = \theta + ib$ $(b < \alpha_1)$ back to the neighbouring point. This alters the branch of the function, and gives us there

$$R = -\sqrt{2rr'(\cosh\alpha_1 - \cosh b)}.$$

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* *Cf.* Sommerfeld, *Math. Ann.*, Bd. XLVII., p. 352.

Then, proceeding through the set of values $\alpha = \theta,\ \theta+\pi,\ \theta+\pi+ib,\ \theta+\pi+i\infty$, we find that in the R-plane the point describes the negative part of the axis of real quantities. Thus the path (p, q, r, s, t, u, v) in the α-plane of Fig. 6 corresponds to the real axis in the R-plane. We should find a similar correspondence from the image of this path in the real axis of the α-plane, and the position taken up by either when instead of θ we have $\theta \pm 2m\pi$. Thus we see that on crossing directly, *i.e.*, without the loop, from one side to the other of any part of these lines in the α-plane, we cross from one side to the other of the real axis of the R-plane, and that without a jump; in other words, we pass from a value of R with an infinitesimal positive or negative imaginary part to one with an infinitesimal negative or positive imaginary part.

To return to the integral (16),

$$u_0 = \frac{1}{2\pi} \int \frac{e^{-i\kappa R}}{R} \frac{e^{i\alpha}}{e^{i\alpha} - e^{i\theta'}}\, d\alpha.$$

We deform our path as in Fig. 6, which must now be explained. It has already been shown that the path (p, q, r, s, t, u, v), and its

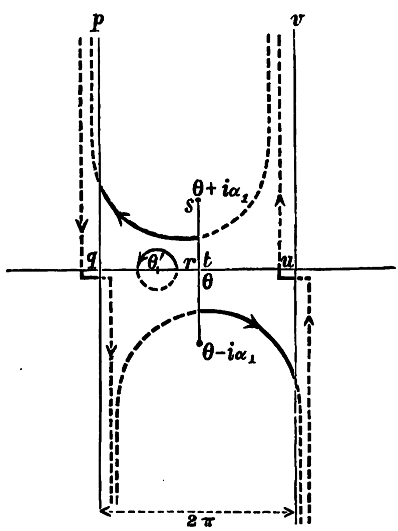

Fig. 6.—Breadth of strip, 2π; deformation of circuit round $\alpha = \theta'$; $|\theta - \theta'| < \pi$; $n = 1$.
In the dotted portions of the path the imaginary part of R is negative.

image in the real axis of a, correspond to the real axis of R; also, that as we pass, in the a-plane, from one side to the other of any part of these broken lines, we pass, in the R-plane, from one side to the other of the axis of real quantities.

Now, from the term $e^{-i\kappa R}$ in our integrand, we must, if we wish to deform the a-path to $a = a \pm i\infty$, ensure that the value of R there has a negative imaginary part.

Starting with the value of R, with positive imaginary part, at a point in the upper part of the figure, our elementary circuit round $a = \theta'$ may be deformed into that composed of the thickly-drawn and dotted lines. The dotted parts denote the portions of the path where the imaginary part of R is negative, and we have made sure that it is negative, by starting with a value of R with positive imaginary part, and remembering that a single crossing of the real axis of R causes the sign of the imaginary part to change. The only places where infinities could arise lie in these portions; so the deformation is permissible.

Making the restriction that the rectilinear portions, those parallel to the imaginary axis, are distant 2π from one another, these portions of our path may be neglected owing to the periodicity of the integrand in a by 2π, and we are left with the identical transformation of u_0 to the integral

$$\frac{1}{2\pi}\int \frac{e^{-i\kappa R}}{R}\,\frac{e^{ia}}{e^{ia}-e^{i\theta'}}\,da,$$

taken over the two curved portions in the a-plane, which we again denote by the path (A).

So far we have had no reason to think of $(\theta-\theta')$ as not contained in $|\,\theta-\theta'\,| < \pi$.

Proceed now to the Multiform Solution.

Consider the function defined by

$$u = \frac{1}{2n\pi}\int \frac{e^{-i\kappa R}}{R}\,\frac{e^{ia/n}}{e^{ia/n}-e^{i\theta'/n}}\,da, \tag{17}$$

the integral being taken over the path (A), corresponding to the current coordinate θ.

This function satisfies the differential equation, since every element of the integral is a solution, and we have excluded infinities. Also the same kind of reasoning that was used in § 2 shows that in the Riemann's space with which we are dealing it is uniform; or, in other words, that it is periodic in θ, and of period $2n\pi$. It also shows

that when $|\theta-\theta'| < \pi$, and (r, θ, z) approaches (r', θ', z'), the function takes the value $\left(\frac{e^{-i\kappa R}}{R}\right)_{R=0}$; that at the underlying points there is no pole, and that at infinity, in all the "examples," the function vanishes. For all these points, and for the general proposition that

$$u_1+u_2+\ldots+u_n = u_0,$$

it is sufficient simply to refer to Fig. 7, drawn for $n = 3$.

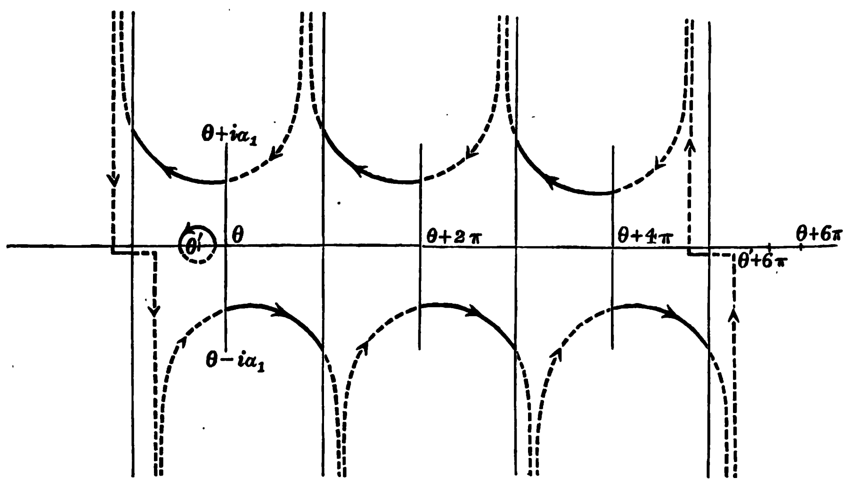

FIG. 7.—Breadth of strip, 2π; deformation of path round $\alpha = \theta'$; $|\theta-\theta'| < \pi$; $n = 3$.
In the dotted portions of the path the imaginary part of R is negative.

To sum up, the function u defined by (17), taken over the proper path (A), corresponding to the θ involved, has the following properties :—

(i.) *It satisfies the equation* $\nabla^2 u+\kappa^2 u = 0$.

(ii.) *It is uniform in the n-fold Riemann's space considered; in other words, it is periodic in* θ *and of period* $2n\pi$.

(iii.) *For all finite values of* (r, θ, z) *it is finite and continuous, unless in the point* (r', θ', z'), *where it possesses a simple pole.*

(iv.) *It vanishes at infinity in all the examples of the Riemann's space.*

(v.) *The n values at corresponding points satisfy the condition*

$$u_1+u_2+\ldots+u_n = u_0.$$

Evaluation of u for $n = 2$.

There would be no difficulty in evaluating u for any value of n. We should need to break up the range into n parts

$$|\,\theta-\theta'\,| < \pi, \quad \pi < |\,\theta-\theta'\,| < 3\pi, \text{ \&c.},$$

and we should obtain integrals for the function in each of these divisions.

It is important for the physical application to find these values for $n = 2$.

We have $$u_1+u_2 = u_0.$$

Also, as before, we are able to deform the path of u_2 into the two lines $\theta+\pi$, $\theta+3\pi$, and we find

$$u_2 = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{1}{\cos\frac{1}{2}(\theta-\theta'+ib)}\,db, \tag{18}$$

i.e.,

$$u_2 = \frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta')\int_0^{\infty} \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b}\,db. \tag{19}$$

This is the value of u at $\theta+2\pi$ when $|\,\theta-\theta'\,| < \pi$, so that, in the second example, we have, for u at (θ),

$$u = -\frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta')\int_0^{\infty} \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b}\,db, \tag{20}$$

while in the first

$$u = u_0-\frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta')\int_0^{\infty} \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b}\,db. \tag{21}$$

At first it would appear that there is a discontinuity here at the passage from one space to the other. The following consideration shows that this is not so.

At $\theta = \pi+\theta'-\epsilon$ (ϵ a small positive quantity)

$$u = u_0-\frac{1}{\pi}\sin\frac{\epsilon}{2}\int_0^{\infty} \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{-\cos\epsilon+\cosh b}\,db. \tag{22}$$

At $\theta = \pi + \theta' + \epsilon$,.

$$u = \frac{1}{\pi} \sin \frac{\epsilon}{2} \int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1 + \cosh b)}}}{\sqrt{2rr'(\cosh a_1 + \cosh b)}} \frac{\cosh \frac{1}{2}b}{-\cos\epsilon + \cosh b} db. \tag{23}$$

There must be a discontinuity unless

$$\mathrm{Lt}_{\epsilon=0} \left(\frac{1}{\pi} \sin \frac{\epsilon}{2} \int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1 + \cosh b)}}}{\sqrt{2rr'(\cosh a_1 + \cosh b)}} \frac{\cosh \frac{1}{2}b}{-\cos\epsilon + \cosh b} db \right) = \frac{1}{2} \frac{e^{i\kappa\sqrt{2rr'(\cosh a_1 + 1)}}}{\sqrt{2rr'(\cosh a_1 + 1)}}.*$$

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* For the following discussion of this integral I am indebted to Prof. Gibson, of the Technical College, Glasgow.

By the substitution used in the text we reduce the expression to

$$\int_0^\infty \phi(x) \frac{dx}{x^2+1},$$

where $$\phi(x) = \frac{e^{-i\kappa\sqrt{4rr'(\cosh^2 \frac{1}{2}a_1 + x^2 \sin^2 \frac{1}{2}\epsilon)}}}{\sqrt{4rr'(\cosh^2 \frac{1}{2}a_1 + x^2 \sin^2 \frac{1}{2}\epsilon)}}.$$

Now choose m so that $\tan^{-1} m \leqq \frac{1}{2}\pi - \epsilon_1$.

Since the integral is convergent, we can choose m, n $(m < n)$ so large that

$$\int_m^n \phi(x) \frac{dx}{x^2+1} < \epsilon_2.$$

If the previous value of m is not large enough to secure this, let it be increased till it does satisfy this condition.

Then $\tan^{-1} m \leqq \frac{1}{2}\pi - \epsilon_1$, would hold *a fortiori*. Hence we have

$$\int_0^n \phi(x) \frac{dx}{x^2+1} = \int_0^m \phi(x) \frac{dx}{x^2+1} + \int_m^n \phi(x) \frac{dx}{x^2+1}$$
$$= \phi(0) \tan^{-1} m + \int_0^m [\phi(x) - \phi(0)] \frac{dx}{x^2+1} + \int_m^n \phi(x) \frac{dx}{x^2+1}.$$

But we may choose ϵ so that $|\phi(x) - \phi(0)| < \epsilon_3$. This involves that $x^2 \sin^2 \frac{1}{2}\epsilon$ be very small. Then

$$\int_0^n \phi(x) \frac{dx}{x^2+1} - \tfrac{1}{2}\pi\,\phi(0) \leqq -\epsilon_1 \phi(0) + \epsilon_2 + \epsilon_3 (\tfrac{1}{2}\pi - \epsilon_1).$$

But $$\int_0^\infty \phi(x) \frac{dx}{x^2+1} - \int_0^n \phi(x) \frac{dx}{x^2+1}$$

is less than any assignable quantity; therefore we have found that

$$\int_0^\infty \phi(x) \frac{dx}{x^2+1} = \tfrac{1}{2}\pi\,\phi(0).$$

It is obvious that m may be taken at once large enough to satisfy all the conditions required for m and n.

Finally, ϵ may be chosen such that $|\phi(x) - \phi(0)| < \epsilon_3$, $0 \leqq x \leqq m$.

Thus the limit$_{\epsilon=0}$ of $\sin \frac{1}{2}\epsilon \int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1 + \cosh b)}}}{\sqrt{2rr'(\cosh a_1 + \cosh b)}} \frac{\cosh \frac{1}{2}b}{-\cos\epsilon + \cosh b} db$ is clearly $\frac{1}{2}\pi\,\phi(0)$.

As a rough proof of this identity, which the physical interpretation of the problem renders necessary, we might adduce the following:—

Put $\sinh \frac{1}{2}b = x \sin \frac{1}{2}\epsilon$, which is possible, as we want the limit of the integral when $\epsilon = 0$, not the value when $\epsilon = 0$.

We then find

$$\sin \frac{\epsilon}{2} \int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1 + \cosh b)}}}{\sqrt{2rr'(\cosh a_1 + \cosh b)}} \frac{\cosh \frac{1}{2}b}{-\cos\epsilon + \cosh b}\, db$$

$$= \int_0^\infty \frac{e^{-\kappa\sqrt{4rr'(\cosh^2 \frac{1}{2}a_1 + x^2 \sin^2 \frac{1}{2}\epsilon)}}}{\sqrt{4rr'(\cosh^2 \frac{1}{2}a_1 + x^2 \sin^2 \frac{1}{2}\epsilon)}} \frac{dx}{x^2+1}.$$

If now we let x approach infinity in such a way that always, in the limit, $x^2 \sin^2 \frac{1}{2}\epsilon$ may be neglected, this gives in the limit

$$\sin \frac{\epsilon}{2} \int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1 + \cosh b)}}}{\sqrt{2rr'(\cosh a_1 + \cosh b)}} \frac{\cosh \frac{1}{2}b}{-\cos\epsilon + \cos(\theta - \theta')}\, db$$

$$= \frac{\pi}{2} \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1 + 1)}}}{\sqrt{2rr_1(\cosh a_1 + 1)}}. \quad (24)$$

Therefore we find that, at the division* between the spaces, the two values of u take the same value $\frac{1}{2}u_0$. We should find a corresponding coincidence at the branch-line in § 2, and in those which follow.

Our solution, then, is one which, in the region contained in the two complete revolutions of θ, from $-(\pi-\theta')$ to $(3\pi+\theta')$, has but one pole, and that at (r', θ', z'). From the periodicity of u by 4π, we may now remove this restriction on the range of θ, and may take it from -2π to $+2\pi$, provided we are careful to use the proper values to be assigned to u at the different points.

It is easy to see that our function has to be considered in three divisions:

$$-2\pi < \theta < -(\pi - \theta'),$$

$$-(\pi-\theta') < \theta < (\pi + \theta'),$$

$$(\pi + \theta') < \theta < 2\pi.$$

---

* This is the branch-membrane (*Verzweigungsmembran*).

In the first

$$u = -\frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta')\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cosh b+\cos(\theta-\theta')}\,db. \tag{25}$$

In the second

$$u = u_0-\frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta')\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cosh b+\cos(\theta-\theta')}\,db. \tag{26}$$

In the third it takes the same form as in the first.

5. *Application to the Theory of Sound.—The Problem of a Source of Sound in an Infinite Medium containing a Fixed Thin Rigid Semi-infinite Plane bounded by a Straight Edge.*

Using cylindrical coordinates, take the plane as given by $\theta = 0$, its edge by the axis of z, and the position of the source by the coordinates $(r', \theta', 0)$.

Then our physical problem quickly reduces to the solution of the equation

$$\nabla^2 u + \kappa^2 u = 0,$$

under the following conditions :—

(i.) $0<\theta<2\pi$; *u is to be finite and continuous for finite values of* (r, z) *except at the point* $(r', \theta', 0)$, *where it is to take the form* $\frac{e^{-i\kappa R}}{R}$, *when* $R = 0$.

(ii.) *It is to be zero at infinity.*

(iii.) $\frac{1}{r}\frac{\partial u}{\partial \theta}$ *is to vanish at* $\theta = 0$ *and* $\theta = 2\pi$.

To obtain this solution we have only to take into consideration the two-fold Riemann's space, with the axis of z as branch-line, and the plane $\theta = 0$ as branch-membrane.

We put poles at $(r', \theta', 0)$ *and* $(r', -\theta', 0)$, *and take the physical space as defined by*

$$0 < \theta < 2\pi.$$

Thus $$\bar{u} = u(\theta') + u(-\theta') \tag{27}$$

satisfies all the conditions of the problem.

As remarked above, care has to be taken to choose the proper values for u in this region. The complete revolution is divided into three portions. From $\theta = 0$ to $\theta = (\pi - \theta')$ both values are those in the first space, namely, those given by u_1. From $\theta = (\pi - \theta')$ to $(\pi + \theta')$, we take u_1 for $u\,(\theta')$, and u_2 for $u\,(-\theta')$. From $\theta = (\pi + \theta')$ to 2π, both values are in the second space.

Hence our solution takes the following forms :—

$$
\begin{aligned}
\text{(A)}\quad u = {} & \frac{e^{-i\kappa\sqrt{2rr'[\cosh a_1 - \cos(\theta-\theta')]}}}{\sqrt{2rr'[\cosh a_1 - \cos(\theta-\theta')]}} \\
& - \frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta')\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b}\,db \\
& + \frac{e^{-i\kappa\sqrt{2rr'[\cosh a_1 - \cos(\theta+\theta')]}}}{\sqrt{2rr'[\cosh a_1 - \cos(\theta+\theta')]}} \\
& - \frac{1}{\pi}\cos\tfrac{1}{2}(\theta+\theta')\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cos(\theta+\theta')+\cosh b}\,db\,; \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
\text{(B)}\quad u = {} & \frac{e^{-i\kappa\sqrt{2rr'[\cosh a_1 - \cos(\theta-\theta')]}}}{\sqrt{2rr'[\cosh a_1 - \cos(\theta-\theta')]}} \\
& - \frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta')\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b}\,db \\
& - \frac{1}{\pi}\cos\tfrac{1}{2}(\theta+\theta')\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cos(\theta+\theta')+\cosh b}\,db\,; \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
\text{(C)}\quad u = {} & -\frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta')\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b}\,db \\
& - \frac{1}{\pi}\cos\tfrac{1}{2}(\theta+\theta')\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr\,(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cos(\theta+\theta')+\cosh b}\,db. \qquad (30)
\end{aligned}
$$

It is easy to see that, at $\theta = 0$ and $\theta = 2\pi$,

$$\frac{\partial u}{\partial \theta} = 0, \quad 0 < r < \infty.$$

The only information we have at the outset from these integrals is that at $(\pi - \theta')$ the part of the disturbance due to the image is the same as if we had at $(2\pi - \theta')$ a source of half the strength. This is deduced from our work above on the continuity of the two expressions. Similarly, that at $(\pi + \theta')$ the effect of the disturbance

due to the source at θ' seems diminished to half, while we have the addition of terms which, from the analogy with what happens in the two-dimensional problem, we might suppose due to sources distributed along the edge of the obstacle. On this analogy the terms in (C) will give the diffracted sound waves.

We can, however, find approximations to the values of our function in the different regions of Fig. 8 as we move away from the origin

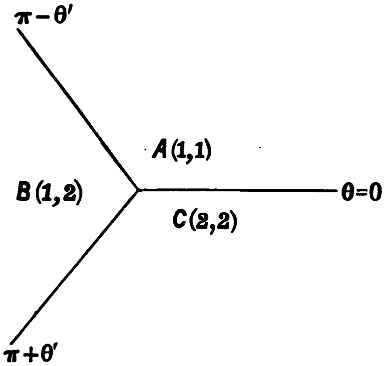

FIG. 8.

and off to infinity. These approximations mark out for us, in some degree, the circumstances of the motion.

Approximation to the Value of u at Infinity.

To obtain this approximation it is necessary to examine the integral

$$\cos\tfrac{1}{2}(\theta-\theta')\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b}\,db.$$

Substitute $\qquad \sinh\tfrac{1}{2}b = x\cos\tfrac{1}{2}(\theta-\theta'),$

and we obtain

$$\int_0^\infty \frac{e^{-i\kappa\sqrt{4rr'[\cosh^2\frac{1}{2}a_1+\cos^2\frac{1}{2}(\theta-\theta')x^2]}}}{\sqrt{4rr'[\cosh^2\frac{1}{2}a_1+\cos^2\frac{1}{2}(\theta-\theta')x^2]}}\,\frac{dx}{x^2+1},$$

i.e., $$\int_0^\infty \phi(x)\frac{dx}{x^2+1},$$

where $$\phi(x) = \frac{e^{-i\kappa\sqrt{(r+r')^2+z^2+4rr'\cos^2\frac{1}{2}(\theta-\theta')x^2}}}{\sqrt{(r+r')^2+z^2+4rr'\cos^2\frac{1}{2}(\theta-\theta')x^2}}.$$

Now consider $$\int_0^m \phi(x)\frac{dx}{x^2+1},$$

$$\int_0^m \phi(x)\frac{dx}{x^2+1} = \phi(0)\int_0^m \frac{dx}{x^2+1} + \int_0^m \left[\phi(x)-\phi(0)\right]\frac{dx}{x^2+1}.$$

Let us choose the infinity of m so that when r, z are infinite $\phi(x)-\phi(0)$ may be taken less than any assignable quantity (this involves rm^2 being negligible in comparison with r^2+z^2). Then we have

$$\mathrm{Lt}_{\substack{m=\infty\\ r=\infty\\ z=\infty}} \int_0^m \phi(x)\frac{dx}{x^2+1} = \tfrac{1}{2}\pi\,\phi(0). \tag{31}$$

The same result follows from the term

$$\cos\tfrac{1}{2}(\theta-\theta')\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cos(\theta+\theta')+\cosh b}\,db.$$

Therefore we see that, when we proceed to a great distance from the pole, the disturbance in (A) is the same as that due to a source at $(r', \theta', 0)$, another at $(r', -\theta', 0)$, and a sink at the pole. In (B) we have the remarkable fact that to our approximation the two latter portions of our expression for u disappear; since $\cos\frac{1}{2}(\theta+\theta')$ is negative, and our integral

$$\cos\tfrac{1}{2}(\theta+\theta')\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cos(\theta+\theta')+\cosh b}\,db$$

becomes, on substituting $\sinh\frac{1}{2}b = -x\cos\frac{1}{2}(\theta+\theta')$,

$$-\int_0^\infty \phi(x)\frac{dx}{x^2+1}.$$

Thus we are left with the part of the disturbance due to the original source alone. In (C) we trace our disturbance to a source of the same strength at the pole.

6. *The Corresponding Problem in Two Dimensions.—Multiform Solutions of the Equation* $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\kappa^2 u = 0$, *with Infinity.*

Although the results in the two-dimensional case are not obtained in a workable form owing to the necessity for introducing Bessel's Functions into the integrals, it is interesting to examine the question from the pure mathematical point of view. We shall obtain results which contain the solution of the problem when we have a source of

sound in two dimensions in space bounded by the semi-infinite plane with a straight edge.

We propose, then, to discuss the solution of the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\kappa^2 u=0,$$

which is at (x', y') infinite as $\log(R)$, when $R=0$.

Following the method already illustrated, we proceed from the simplest uniform solution of our equation with an infinity as required.

For this case, *i.e.*, in the physical interpretation, when we have a symmetrical disturbance diverging from the pole in an infinite space, our solution is given by Rayleigh, *Theory of Sound*, Vol. II., § 341, where that problem is fully discussed. The solution may be written in either of the two following ways :—

$$Y_0(\kappa r)=\left(\gamma+\log\frac{i\kappa r}{2}\right)\left(1-\frac{\kappa^2r^2}{2^2}+\frac{\kappa^4r^4}{2^2.4^2}-\&c.\right)+\frac{\kappa^2r^2}{2^2}S_1-\frac{\kappa^4r^4}{2^2.4^2}S_2+\&c., \tag{32}$$

where $\gamma=$ Euler's constant, and

$$S_m=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m};$$

or $$Y_0(\kappa r)=-\left(\frac{\pi}{2i\kappa r}\right)^{\frac{1}{2}}e^{-i\kappa r}\left(1-\frac{1^2}{1.8i\kappa r}+\frac{1^2.3^2}{1.2(8i\kappa r)^2}-\ldots\right). \tag{33}$$

Referring to Gray and Mathews' *Treatise on Bessel's Functions*, p. 22, (50), we find the proof that this value of Y_0 may be written as

$$\left(\gamma+\log\frac{i\kappa r}{2}\right)J_0+4\left(\frac{J_2}{2}-\frac{J_4}{4}+\ldots\right), \tag{34}$$

and we see that this is related to the solution used by J. J. Thomson in his "Recent Researches" (the sign of C being corrected), and by Sommerfeld in *Math. Ann.*, Bd. XLVII., p. 327, by the equation

$$Y_0(x)=-U_0(x)=\frac{i\pi}{2}J_0(x)-K_0(x). \tag{35}$$

Suppose the pole at (r', θ'), and we must change our solution to $Y_0(\kappa R)$, where

$$R=\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}.$$

Introduce, as before, the complex variable α, and we have the identical transformation

$$Y_0(\kappa R) = \frac{1}{2\pi}\int Y_0(\kappa R')\frac{e^{i\alpha}}{e^{i\alpha}-e^{i\theta'}}\,d\alpha \tag{36}$$

$$\left[R'^2 = r^2+r'^2-2rr'\cos(\alpha-\theta),\ \text{putting } \alpha \text{ for } \theta' \text{ above}\right],$$

the integral being taken round a small circuit in the α-plane enclosing $\alpha = \theta'$, and no other singularity or branch-point of the integrand.

Before discussing the possible deformations of our path we must examine these critical points.

From the equation

$$Y_0(\kappa R') = \left(\gamma+\log\frac{i\kappa R'}{2}\right)\left(1-\frac{\kappa^2R'^2}{2^2}+\frac{\kappa^4R'^4}{2^2.4^2}-\&c.\right)$$
$$+\frac{\kappa^2R'^2}{2^2}S_1-\frac{\kappa^4R'^4}{2^2.4^2}S_2+\&c.,$$

it is evident that the branch-points are given by those of $R'=0$, *i.e.*, by $\alpha=\theta+2m\pi\pm i\alpha_1$, where $\cosh\alpha_1 = \dfrac{r^2+r'^2}{2rr'}$.

In considering the behaviour of $Y_0(\kappa R')$ at infinity we take the second of the forms given. It follows that a condition necessary for a possible deformation of the path to $\alpha = a\pm ib$. $(b=\infty)$ is that the imaginary part of R' there be negative. Hence our work is absolutely analogous to that in the former section. We are able to deform the path as given in Fig. 6, and to take as our multiform solution

$$u = \frac{1}{2n\pi}\int Y_0(\kappa R')\frac{e^{i\alpha/n}}{e^{i\alpha/n}-e^{i\theta'/n}}\,d\alpha \tag{37}$$

over the path (A) corresponding to the current coordinate (θ).

By means of a discussion similar to that on pp. 137–140, we should find that this function has the following properties :—

(i.) *It satisfies the differential equation*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\kappa^2 u = 0.$$

(ii.) *It is uniform on our n-sheeted Riemann's surface ; in other words, it is periodic in θ, and of period $2n\pi$.*

(iii.) *It is finite and continuous for all finite values of* (x, y), *except in the point* (x', y'), *where it has a simple pole.*

(iv.) *It vanishes at infinity:*

(v.) *The values at the* n *corresponding points of the Riemann's surface satisfy the equation*

$$u_1+u_2+\ldots+u_n = Y_0(\kappa R),$$

where $$R = \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}.$$

Just as before, we could obtain integrals giving the values of u for any assigned integer n.

For the application to the physical problem of a line source parallel to a semi-infinite rigid thin plane, we require the value for $n = 2$, so that the period of the function may be 4π.

We obtain the following expressions for u on the first and second sheets respectively:—

$$u_1 = Y_0\left[\kappa\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}\right]$$
$$-\frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta')\int_0^\infty Y_0\left[\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}\right]\frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b}\,db, \quad (38)$$

and

$$u_2 = -\frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta')\int_0^\infty Y_0\left[\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}\right]\frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b}\,db. \quad (39)$$

7. *Application to the Theory of Sound.—The Problem in Two Dimensions of a Source outside a Semi-infinite Thin Rigid Plane bounded by a Straight Edge.*

Taking the physical space as defined by $0<\theta<2\pi$, the source as at (r', θ'), and the obstacle as $\theta = 0$, $0<r<\infty$, we obtain the required solution from the function found in the last section.

This solution is $$\bar{u} = u(\theta')+u(-\theta'),$$

and in evaluating it we have to break up the area into the three portions

$$0 < \theta < \pi-\theta', \quad \text{(A)}$$

$$\pi-\theta' < \theta < \pi+\theta', \quad \text{(B)}$$

$$\pi+\theta' < \theta < 2\pi. \quad \text{(C)}$$

In these we have the following results :—

(A) $u = Y_0(\kappa R) - \frac{1}{\pi}\cos\frac{1}{2}(\theta-\theta')\int_0^\infty Y_0[\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}]\ \frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b}\,db$

$$+ Y_0[\kappa\sqrt{r^2+r'^2-2rr'\cos(\theta+\theta')}]$$

$$-\frac{1}{\pi}\cos\frac{1}{2}(\theta+\theta')\int_0^\infty Y_0[\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}]\ \frac{\cosh\frac{1}{2}b}{\cos(\theta+\theta')+\cosh b}\,db, \quad (40)$$

(B) $u = Y_0(\kappa R) - \frac{1}{\pi}\cos\frac{1}{2}(\theta-\theta')\int_0^\infty Y_0[\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}]\ \frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b}\,db$

$$-\frac{1}{\pi}\cos\frac{1}{2}(\theta+\theta')\int_0^\infty Y_0[\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}]\ \frac{\cosh\frac{1}{2}b}{\cos(\theta+\theta)+\cosh b}\,db, \quad (41)$$

(C) u = the last two expressions of (B). (42)

Hence, from analogy with what we have found above, we may say that in (A) there exist incident, reflected, and diffracted waves; in (B) incident and diffracted; in (C) diffracted, only; and that they are represented by the respective parts of the above expressions.

8. *Multiform Solution of the Partial Differential Equation of the Theory of the Conduction of Heat in a Body of Uniform Conductivity.— Two-Dimensional Case.*

So far we have been considering the equation which meets us in oscillatory motion, be it in the vibrations of sound, light, or electricity. It is a much simpler problem, though perhaps not so interesting, to examine the corresponding solutions of the equation which forms the basis of the mathematical theory of the conduction of heat, namely,

$$\frac{\partial u}{\partial t} = \kappa\nabla^2 u.$$

As in the potential theory use has been made of the particular solution $\frac{1}{r}$ and in that of sound of $\frac{e^{-i\kappa r}}{r}$ and $e^{i\kappa r\cos(\theta-\theta')}$, so here we start from the distribution of temperature in an infinite solid of uniform conductivity, due to a unit quantity of heat, placed at the time $t=0$ at the point (x', y', z') and left to diffuse.

The temperature at (x, y, z) at time t is given by

$$u = \frac{1}{2^3(\pi\kappa t)^{\frac{3}{2}}}\,e^{-[(x-x')^2+(y-y')^2+(z-z')^2]/4\kappa t} \quad (43)$$

This synthetical method of dealing with the subject has been used by Kelvin,* Hobson,† Bryan,‡ and Sommerfeld.§

In this section, and in those which follow, I propose to find solutions suitable for the application of this method to cases where the ordinary image theory fails; that is, to those where we must imagine not the ordinary, but a Riemann's, space to be that in which we desire a solution of the equation.

We begin with the two-dimensional problem, and start from the solution

$$u_0 = \frac{1}{t}\, e^{-[(x-x')^2+(y-y')^2]/4\kappa t} = \frac{1}{t}\, e^{-[r^2+r'^2-2rr'\cos(\theta-\theta')]/4\kappa t}, \tag{44}$$

which differs by a constant multiplier from the temperature due to a unit source of heat.

Introduce the complex variable α, and we have the identical transformation

$$u_0 = \frac{1}{2\pi}\int \frac{e^{-[r^2+r'^2-2rr'\cos(\alpha-\theta)]/4\kappa t}}{t}\,\frac{e^{i\alpha}}{e^{i\alpha}-e^{i\theta'}}\,d\alpha, \tag{45}$$

the integral being taken over a path in the α-plane, enclosing $\alpha = \theta'$, and no other singularity of the integrand.

On these conditions

$$u_0 = \frac{1}{2\pi t}\, e^{-(r^2+r'^2)/4\kappa t}\int e^{rr'/2\kappa t\cos(\alpha-\theta)}\,\frac{e^{i\alpha}}{e^{i\alpha}-e^{i\theta'}}\,d\alpha. \tag{46}$$

The only ways in which singularities can occur are from the poles $\alpha = 2m\pi+\theta'$, and the infinities of $e^{rr'/2\kappa t\cos(\alpha-\theta)}$. On putting $\alpha = a+ib$, since

$$\cos(\alpha-\theta) = \cos(a-\theta)\cosh b - i\sin(a-\theta)\sinh b,$$

we see that, when $b = \pm\infty$, $\cos(a-\theta)$ must be negative, or an infinite value will be given to the integrand.

Hence, in deforming the path to $b = \pm\infty$, we must take care to have α in such a region that $\cos(a-\theta)$ be negative.

The shaded portions of Fig. 9 represent such parts of the α-plane, and, taking $|\theta-\theta'| < \pi$, the circuit round $\alpha = \theta'$ may be deformed

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* *Math. and Physical Papers*, Vol. II., LXXII., "Compendium of the Fourier Mathematics."

† *Proc. Lond. Math. Soc.*, Vol. XIX., "Synthetic Solutions in the Theory of Heat."

‡ *Proc. Lond. Math. Soc.*, Vol. XIX., "An Application of the Method of Images to the Theory of Heat."

§ *Math. Ann.*, Bd. XLV., "Zur analytischen Theorie der Wärme-leitung."

into that there given;* the new path being composed of two curved parts extending to infinity and two rectilinear parts.

These rectilinear parts—dotted in the figure—are supposed drawn at distance 2π from one another, and therefore the portions of the integral contributed by these, taken in opposite directions, disappear owing to the periodicity of the integrand. We are left with the integral over the two curved portions, which we, as before, denote by the integral over the path (A). It is to be noted that, as the function

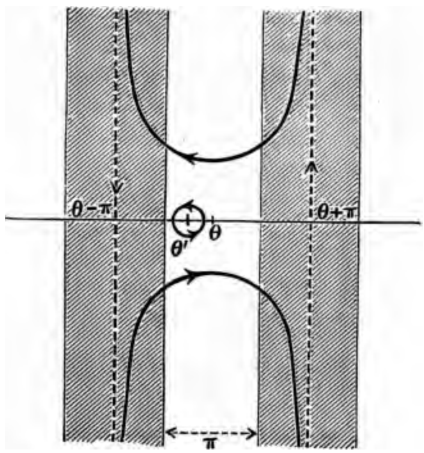

FIG. 9.—Breadth of strip, π; deformation of circuit round $\alpha = \theta'$;
$|\theta-\theta'| < \pi$; $n = 1$.
In the shaded portions $\cos(\alpha-\theta)$ has a negative real part.

to be integrated is uniform and has no branch-points, the question of the deformation of the path is much simpler here than in the former problems.

We now obtain the Multiform Solution.

Consider the integral

$$u = \frac{1}{2n\pi}\frac{e^{-(r^2+r'^2)/4\kappa t}}{t}\int e^{rr'/2\kappa t\cos(\alpha-\theta)}\frac{e^{i\alpha/n}}{e^{i\alpha/n}-e^{i\theta'/n}}\,d\alpha, \tag{47}$$

taken over the path (A), corresponding to the value of the current coordinate θ; we have given up the restriction

$$|\theta-\theta'| < \pi.$$

* It is not necessary that θ' lie on the unshaded portion. It must lie, in the first instance, between the two lines $\alpha = \theta\pm\pi$.

This function u is a solution of our differential equation, as every element of the integral is a solution, and infinite values are excluded from the path.

It is also periodic in θ and of period $2n\pi$; or, in other words, on the n-sheeted Riemann's surface, with the line from the origin to infinity in the direction $(\pi+\theta')$ as branch-section, the function is uniform.

The proof of this is exactly similar to that of the former sections. Changing the value of θ by $2n\pi$ simply moves the path through a distance $2n\pi$. The value of the integrand at each point of the new path is the same as the value at the corresponding point of the old, because of its periodicity by $2n\pi$ in α. Hence the above result.

When $t=0$, the value of u vanishes, unless at the point (r', θ'), where it takes the form

$$\mathrm{Lt}_{\substack{r=r' \\ \theta=\theta' \\ t=0}} \left(\frac{e^{-[r^2+r'^2-2rr'\cos(\theta-\theta')]/4\kappa t}}{t}\right).$$

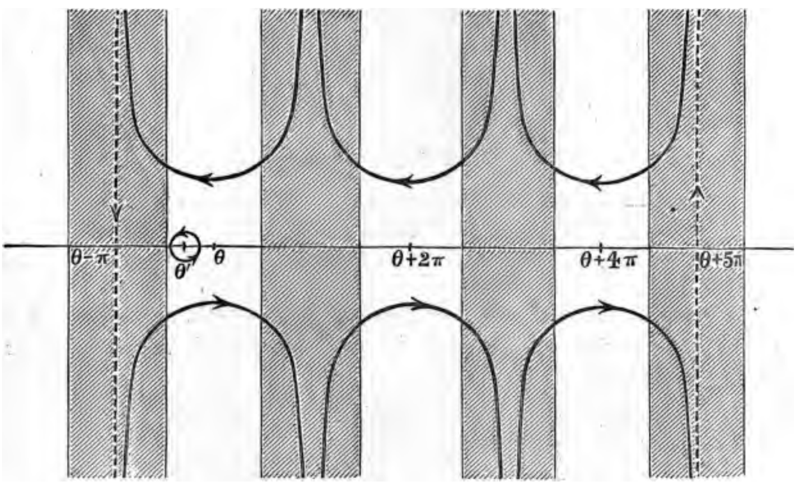

FIG. 10.—Breadth of strip, π; deformation of circuit round $\alpha=\theta'$; $|\theta-\theta'|<\pi$; $n=3$.

To prove this it is simplest to consider Fig. 10, where we have taken $n=3$, and have drawn the curves for a point θ on the first sheet, *i.e.*, when $|\theta-\theta'|<\pi$, and for the underlying points on the other two sheets, *i.e.*, for the points $\theta+2\pi$, $\theta+4\pi$. For points on the second and third sheets our path (A) can be replaced by

the rectilinear path over the two lines distant by 2π (dotted in figure), and these, being completely in the shaded portion, vanish when $t = 0$, since every element of the integrand then vanishes. For points on the first sheet we have, in addition to the straight lines, to take the circuit round the pole $\alpha = \theta'$; and hence in the first sheet

$$u = u_0 = \frac{e^{-[r^2+r'^2-2rr'\cos(\theta-\theta')]/4\kappa t}}{t}, \quad \text{when } t = 0. \tag{48}$$

This vanishes, unless at the point (r', θ').

Hence we see that, for finite values of r, the integral is zero, for $t = 0$, unless at the point (r', θ'), when it takes the form

$$\mathrm{Lt}_{\substack{t=0\\R=0}} \left(\frac{e^{-(R^2/4\kappa t)}}{t}\right).$$

The term $e^{-(r^2/4\kappa t)}$ causes the integral to vanish at infinity on all the sheets.

Finally, we have the relation between the values of u at underlying points on the surface at any time. This is proved, just as before, from Fig. 10, and is expressed by the equation

$$u_1 + u_2 + \ldots + u_n = u_0.$$

To sum up, the function u just found has the following properties :—

(i.) *It is a solution of the equation*

$$\frac{\partial u}{\partial t} = \kappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

(ii.) *It is uniform on the n-sheeted Riemann's surface considered; in other words, it is of period $2n\pi$ in θ.*

(iii.) *On the first sheet of the surface, i.e., when* $|\theta - \theta'| < \pi$, $u = u_0$, *when* $t = 0$; *on the other sheets* $u = 0$. *At the point* (r', θ'), u *takes the form*

$$\left(\frac{e^{-(R^2/4\kappa t)}}{t}\right)_{\substack{R=0\\t=0}}.$$

(iv.) *It vanishes at infinity on all the sheets.*

(v.) *The values at the corresponding points on the different sheets satisfy the equation*

$$u_1 + u_2 + \ldots + u_n = u_0.$$

Evaluation of u for $n = 2$.

The case to which we desire to apply our multiform solution is that in which $n = 2$.

As before, $$u_1 + u_2 = u_0 = \frac{e^{-[r^2+r'^2-2rr'\cos(\theta-\theta')]/4\kappa t}}{t}.$$

Also we can deform the path (A), for u_2, into the two lines $\alpha = \theta+\pi$, $\alpha = \theta+3\pi$, taken in opposite directions, and we obtain

$$u_2 = \frac{1}{4\pi t} e^{-(r^2+r'^2)/4\kappa t} \int_{-\infty}^{\infty} e^{-(rr'/2\kappa t)\cosh b} \frac{1}{\cos\frac{1}{2}(\theta-\theta'-ib)} db. \qquad (49)$$

Let us write C and c for

$$\frac{1}{\pi t} e^{-(r^2+r'^2)/4\kappa t} \quad \text{and} \quad \frac{rr'}{2\kappa t}.$$

Then $$u_2 = C\cos\tfrac{1}{2}(\theta-\theta') \int_0^{\infty} e^{-c\cosh b} \frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b} db;$$

and

$$\frac{u_2}{u_0} = \frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta') \int_0^{\infty} e^{-c[\cosh b+\cos(\theta-\theta')]} \frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b} db$$

$$= X, \text{ say};$$

therefore

$$\frac{\partial X}{\partial r} = -\frac{c}{r\pi}\cos\tfrac{1}{2}(\theta-\theta')\, e^{-2c\cos^2\frac{1}{2}(\theta-\theta')} \int_0^{\infty} e^{-2c\sinh^2\frac{1}{2}b}\cosh\tfrac{1}{2}b\, db,$$

i.e., $$\frac{\partial X}{\partial r} = -\tfrac{1}{2}\sqrt{\frac{r'}{\pi r\kappa t}}\cos\tfrac{1}{2}(\theta-\theta')\, e^{-(rr'/2\kappa t)\cos^2\frac{1}{2}(\theta-\theta')};$$

therefore $$\frac{\partial X}{\partial r} = -\frac{1}{\sqrt{\pi}}\frac{\partial}{\partial r}\int_0^{\sqrt{rr'/\kappa t}\cos\frac{1}{2}(\theta-\theta')} e^{-\lambda^2} d\lambda,$$

therefore $$X = -\frac{1}{\sqrt{\pi}}\int_0^{T} e^{-\lambda^2} d\lambda + X_0,$$

where $$T = \sqrt{\frac{rr'}{\kappa t}}\cos\tfrac{1}{2}(\theta-\theta'),$$

and X_0 is the value of X when $r = 0$.

It is easy to show that $X_0 = \frac{1}{2}$.

Hence
$$u_2 = u_0\left(\tfrac{1}{2} - \frac{1}{\sqrt{\pi}}\int_0^T e^{-\lambda^2}\,d\lambda\right)$$
$$= \frac{1}{\sqrt{\pi}}\,u_0 \int_{-\infty}^{-T} e^{-\lambda^2}\,d\lambda; \tag{50}$$

and
$$u_1 = u_2 - u_0$$
$$= \frac{1}{\sqrt{\pi}}\,u_0 \int_{-\infty}^{T} e^{-\lambda^2}\,d\lambda. \tag{51}$$

Remembering that this expression for u_2 is that for u at the point $(r, \theta+2\pi)$, when $|\theta-\theta'| < \pi$, we obtain for u on the second sheet at the point (r, θ) the same form

$$\frac{1}{\sqrt{\pi}}\,u_0 \int_{-\infty}^{T} e^{-\lambda^2}\,d\lambda$$

as for u on the first sheet.

We have thus found a function

$$u = \frac{1}{\sqrt{\pi}}\,\frac{e^{-[r^2+r'^2-2rr'\cos(\theta-\theta')]/4\kappa t}}{t}\int_{-\infty}^{\sqrt{rr'/\kappa t}\,\cos\frac{1}{2}(\theta-\theta')} e^{-\lambda^2}\,d\lambda, \tag{52}$$

with the following properties :—

(i.) *It is a solution of the equation*

$$\frac{\partial u}{\partial t} = \kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

(ii.) *On the Riemann's surface considered it is uniform ; or, in other words, it is periodic in θ, and of period 4π.*

(iii.) *On this surface it has only the one pole, and that at the point (r', θ'), at which point u takes the value u_0, while at all other points u vanishes for $t = 0$.*

(iv.) *When $r = \infty$, u vanishes.*

Since the function is periodic and of period 4π, there is no reason why we should retain the range

$$-(\pi-\theta') < \theta < (3\pi+\theta').$$

We may take any more suitable one with 4π as its magnitude, and the simplest is

$$-2\pi < \theta < 2\pi.$$

In this region our function u would have but the one pole, and would satisfy the conditions above, care being taken to discriminate between the sheets of the surface; in other words, to choose the proper value of u for the point considered.

9. *Application to the Theory of the Conduction of Heat.—The Problem of an Instantaneous Line Source in an Infinite Body of Uniform Conductivity κ in which there is a Semi-infinite Plane bounded by a Straight Edge: the Plane either* (i.) *kept always at Zero Temperature, or* (ii.) *coated in such a way that no Transference of Heat is possible across it.*

Taking a plane normal to the line as the plane $z=0$, our problem is one in two dimensions. With the origin at the edge, the given plane as $\theta=0$, and the line source passing through (r', θ'), we are able at once to apply the solution of the last section. We consider the physical space as defined by $0<\theta<2\pi$, and we introduce the imaginary space $-2\pi<\theta<0$.

Then
$$\bar{u} = u(\theta') \mp u(-\theta') \quad (0<\theta'<2\pi) \tag{53}$$
are the solutions corresponding to the two cases.

In the space $0<\theta<2\pi$, $\bar{u}$ is zero at time $t=0$ for all values of r, except at the point (r', θ'), where it takes the form
$$\left(\frac{e^{-(R^2/4\kappa t)}}{t}\right)_{\substack{t=0\\R=0}}.$$

Further, at infinity $\bar{u}=0$.

The symmetry of the expression shows us that the boundary conditions are satisfied at $\theta=0$ and $\theta=2\pi$.

This is clear when we note that
$$\bar{u} = \frac{1}{\sqrt{\pi}\, t}\left(e^{-(R^2/4\kappa t)}\int_{-\infty}^{\sqrt{rr'/\kappa t}\cos\frac{1}{2}(\theta-\theta')} e^{-\lambda^2}\,d\lambda \mp e^{-(R'^2/4\kappa t)}\int_{-\infty}^{\sqrt{rr'/\kappa t}\cos\frac{1}{2}(\theta+\theta')} e^{-\lambda^2}\,d\lambda\right), \tag{54}$$
where
$$R^2 = (x-x')^2+(y-y')^2,$$
$$R'^2 = (x-x')^2+(y+y')^2.$$

In the first case $\bar{u}=0$, when $\theta=0$ and $\theta=2\pi$.

In the second $\dfrac{\partial\bar{u}}{\partial\theta}=0$, when $\theta=0$ and $\theta=2\pi$.

The pole in the space $-2\pi < \theta < 0$, does not affect the validity of the result, as we may fix upon one complete revolution about the axis of z as defining absolutely and covering wholly the range of points entering into the space or body considered.

In the paper "On Conduction of Heat" in *Math. Ann.*, Bd. XLV., this result is quoted by Sommerfeld, and he states that it had been obtained by him after a somewhat laborious calculation from the suitable expression in Bessel's functions

$$u = \cos \tfrac{1}{2}n\,(\theta-\theta') \int_{-\infty}^{\infty} e^{-\kappa\lambda^2 t} \sum_{-\infty}^{\infty} J_{\frac{1}{2}n}(\lambda r)\, J_{\frac{1}{2}n}(\lambda r')\,\lambda\, d\lambda. \tag{55}$$

The importance of the method here developed is that, as will be shown immediately, there is no difficulty in at once extending the results to the three-dimensional case. Also it places these problems on the same level with those in sound, light, &c., and the extensions to cases in which the physical conditions are different will find their application at once in the conduction of heat.

10. *Multiform Solution of the Equation* $\dfrac{\partial u}{\partial t} = \kappa \nabla^2 u$.

The work here follows the same lines as in the two-dimensional case.

We start from the particular solution

$$u_0 = \frac{1}{t^{\frac{3}{2}}}\, e^{-[(x-x')^2+(y-y')^2+(z-z')^2]/4\kappa t}; \tag{56}$$

or, in cylindrical coordinates,

$$u_0 = \frac{1}{t^{\frac{3}{2}}}\, e^{-[r^2+r'^2+(z-z')^2-2rr'\cos(\theta-\theta')]/4\kappa t}.$$

Then we obtain the identical transformation

$$u_0 = \frac{1}{2\pi}\, \frac{e^{-[r^2+r'^2+(z-z')^2]/4\kappa t}}{t^{\frac{3}{2}}} \int e^{(rr'/2\kappa t)\cos(a-\theta)}\, \frac{e^{ia}}{e^{ia}-e^{i\theta'}}\, da \tag{57}$$

over a circuit in the a-plane, enclosing $a = \theta'$ and no other singularity of the integrand.

This reduces to the integral over the path (A) of the former section.

To obtain the Multiform Solution, it is only necessary to consider the integral

$$u = \frac{1}{2n\pi}\, \frac{e^{-[r^2+r'^2+(z-z')^2]/4\kappa t}}{t^{\frac{3}{2}}} \int e^{(rr'/2\kappa t)\cos(a-\theta)}\, \frac{e^{ia/n}}{e^{ia/n}-e^{i\theta'/n}}\, da, \tag{58}$$

taken over the path (A) corresponding to the value of θ. This is the multiform solution with a pole at (r', θ', z') in the range $-(\pi-\theta') < \theta < (2n-1)\pi+\theta'$.

Thus we see that the sole difference in our results for the three-dimensional case consists in the introduction of the factors

$$e^{-(z-z')^2/4\kappa t} \quad \text{and} \quad \frac{1}{t^{\frac{3}{2}}}.$$

In the particular case when $n = 2$,

$$u = \frac{1}{\sqrt{\pi}} \frac{e^{-[r^2+r'^2+(z-z')^2-2rr'\cos(\theta-\theta')]/4\kappa t}}{t^{\frac{3}{2}}} \int_{-\infty}^{\sqrt{rr'/\kappa t}\cos\frac{1}{2}(\theta-\theta')} e^{-\lambda^2}\,d\lambda. \qquad (59)$$

11. These physical applications of the multiform solutions found in this paper have been given because of their simplicity and the possibility of testing their agreement with the facts of nature.

The cases in which the planes meet at an angle $\frac{n\pi}{m}$ (n, m positive integers) may be discussed by the same method. Here we should require the n-fold Riemann's surface, or space; or, in other words, our physical space would be defined by one complete revolution round the axis of z, and we should bring to our aid $(n-1)$ imaginary spaces, built up by the successive $(n-1)$ revolutions of the radius vector in the cylindrical coordinate system.

No attempt has been made here to prove the uniqueness of the solutions in the particular cases. This was done for the problems in potential in the often-quoted paper in our *Proceedings*. The physical applications prove that they are unique. An analytical proof I hope to give later.

The next advance in this method ought to be the solution of the problems where the obstacle consists of an infinite plane in which there is a slit with parallel edges; or an infinite plane with parallel edges. The system of bipolar coordinates

$$\rho = \log\left(\frac{r_1}{r_2}\right),$$

$$\phi = \theta_1 - \theta_2$$

gives us a suitable transformation for this case. We have to deal with the integration of our partial differential equations on a Riemann's surface, or space, which has $\phi = 0$ for branch-section, or membrane, and two branch-points, or lines, at the points $\rho = \pm\infty$.

It is obvious that this amounts to defining our physical space by the range

$$0 < \phi < 2\pi,$$

and putting the image in the space defined by

$$-2\pi < \phi < 0.$$

The problem—for the equation of potential—was discussed in Sommerfeld's paper on that equation. [See note by Dr. Sommerfeld, below.] The solutions of the corresponding problems for the equations with which this paper deals at present occupy my attention.

It only requires the discovery of a proper coordinate system to advance our knowledge to the cases examined by the method of series and in approximation by Prof. Lamb, and such a discovery ought to give us, not only exact solutions, but solutions also applicable to three-dimensional work.

The question of the solution of these partial differential equations on other Riemann's surfaces should be a fruitful one also for the pure mathematician, and all these questions which, in the theory of functions, have circled round the potential would enter here for discussion.*

Note by Dr. Sommerfeld to Mr. Carslaw's paper.

Dr. Sommerfeld takes this opportunity of calling attention to an error in his discussion of the problem in potential, where a point charge is placed in the region bounded by an infinite conducting plane, in which there is a slit with parallel edges:—

In den folgenden Zeilen bitte ich ein Versehen berichtigen zu dürfen, welches sich in § 5 meiner in Vol. XXVIII. der *Proc. Lond. Math. Soc.* abgedruckten Arbeit eingeschlichen hat. Ich thue dieses um so lieber, als Herr H. S. Carslaw auf den vorangehenden Seiten zu meiner Freude und auf meine Anregung hin gezeigt hat, dass sich die Methode jener Arbeit in der am Schluss (p. 429) angedeuteten Weise auf andere physikalische Differentialgleichungen ausdehnen lässt.

Der Fehler besteht darin, dass bei Benutzung des p. 421 angegebenen Wertes von R'^2 die Function u aus Gleichung (5), p. 422, zwar allen übrigen Bedingungen des Problems, aber nicht der Differential-

* *Cf.* Pockel's, *Über die partielle Differential-Gleichung* $\nabla^2 u + \kappa^2 u = 0$, pp. 225, 238, 339.

gleichung des Potentials genügt. Um Letzteres zu erreichen, muss man vielmehr nach dem p. 405 genannten Principe den Winkel ϕ' in dem Ausdrucke von R^2 *durchweg* durch die Integrationsvariable α ersetzen, und, dementsprechend, R'^2 folgendermassen definiren:

$$R'^2 = 2\frac{\cos i(\rho-\rho')-\cos(\phi-\alpha)}{(\cos i\rho-\cos\phi)(\cos i\rho'-\cos\alpha)}+(z-z')^2.$$

Gleichzeitig wird es nötig, die Wahl der Function $f(\alpha)$ etwas abzuändern, damit $f(\alpha)/R'$ für $\alpha=\infty$ verschwindet. Man nehme zu dem Zwecke statt der p. 422 angegebenen beiden Werte

$$f(\alpha)=\frac{ie^{i\alpha}}{e^{i\alpha}-e^{i\phi'}}\sqrt{\frac{\cos i\rho'-\cos\phi'}{\cos i\phi'-\cos\alpha}},$$

bez.
$$f(\alpha)=\frac{i}{n}\,\frac{e^{i\alpha/n}}{e^{i\alpha/n}-e^{i\phi'/n}}\sqrt{\frac{\cos i\rho'-\cos\phi'}{\cos i\rho'-\cos\alpha}}.$$

$f(\alpha)$ besitzt dann immer noch die Eigenschaft, für $\alpha=\phi'$ von der ersten Ordnung mit dem Residuum 1 unendlich zu werden. Als Verzweigungspunkte des Integranden kommen ausser $\alpha=\infty$ nur diejenigen Stellen der α-Ebene in Betracht, für welche $R'^2=0$, d. h.,

$$\cos(\phi-\alpha)-\cos i(\rho-\rho')=\tfrac{1}{2}(z-z')^2(\cos i\rho-\cos\phi)(\cos i\rho'-\cos\alpha)$$

wird. Sie sind sehr leicht zu bestimmen, wenn $z-z'=0$; dann haben wir nämlich einfach

$$\alpha=\phi+2k\pi\pm i(\rho-\rho').$$

Im anderen Falle muss man die Gleichung für α auflösen, und erhält

$$\alpha=a+2k\pi\pm ib,$$

wo die Grössen a und b reelle Zahlen bedeuten, die von ϕ, ρ, ρ' und $z-z'$ abhängen.

Die Deformation des Integrationsweges lässt sich darauf gerade so ausführen wie p. 422 angegeben. Der mit W bezeichnete Weg führt, vom Unendlichen ausgehend und dahin zurückkehrend, in einer Schleife um die Verzweigungspunkte $\alpha=a+ib$ und $\alpha=a-ib$ herum.

Die Schlussformel (5) ist hiernach folgendermassen abzuändern:

$$\text{(V)}\quad u=\frac{1}{2\pi n}\int\frac{1}{R'}\sqrt{\frac{\cos i\rho'-\cos\phi'}{\cos i\rho-\cos\alpha}}\,\frac{e^{i\alpha/n}\,d\alpha}{e^{i\alpha/n}-e^{i\phi'/n}}.$$

Die folgenden Bemerkungen über Näherungsformeln in der Nähe der Verzweigungslinien und über die Ausführung der Integration im Falle $n=2$ sind in der pp. 423 und 424 gegebenen Form unmittelbar aufrecht zu halten, wenn man sich auf Punkte der Ebene $z=z'$ beschränkt; im diesem Falle stimmt nämlich die berichtigte Form (V) mit der früher angegebenen (5) genau überein. An der p. 425 beschriebenen Figur, welche sich gerade auf diese Ebene $z=z'$ bezieht, ist daher nichts zu corrigiren.

Ein geringfügiger Schreibfehler, auf den mich Herr Carslaw aufmerksam machte, findet sich ausserdem p. 417. Die Gleichung (3) muss nämlich lauten:

$$v = \frac{2}{\pi R} \arctan \sqrt{\frac{\sigma+\tau}{\sigma-\tau}} - \frac{2}{\pi R'} \arctan \sqrt{\frac{\sigma+\tau'}{\sigma-\tau'}},$$

wobei zur Abkürzung

$$R^2 = r^2+r'^2-2rr' \cos(\phi-\phi')+(z-z')^2,$$

$$R'^2 = r^2+r'^2-2rr' \cos(\phi+\phi')+(z-z')^2$$

gesetzt ist, und wobei σ, τ, τ' die pp. 413 und 417 angegebene Bedeutung haben.

Thursday, January 12th, 1899.

Prof. ELLIOTT, F.R.S., Vice-President; and subsequently Lt.-Col. CUNNINGHAM, R.E., Vice-President, and Dr. HOBSON, F.R.S., in the Chair.

Fourteen members, and a visitor, present.

Prof. Elliott referred, in feeling terms, to the recent death of the Rev. B. Price, F.R.S., who was elected a member of the Society June 26th, 1866.

Dr. Morrice read a paper on "Linear Transformation by Inversions."

Mr. H. M. Macdonald spoke on "The Zeroes of the Bessel Functions" (in continuation of his previous paper on the subject).

Lt.-Col. Cunningham communicated a paper by Mr. D. Biddle, entitled "A Simple Method of Factorizing large Composite Numbers of any unknown Form."

Messrs. Lawrence, Larmor, Hobson, and Western spoke upon one or more of the above papers.

The following papers were communicated, in abstract, by Dr. Hobson, viz.:—

On a Determinant each of whose Elements is the Product of k Factors: Prof. Metzler.

Properties of Hyperspace, in relation to Systems of Forces, the Kinematics of Rigid Bodies, and Clifford's Parallels: Mr. A. N. Whitehead.

On the Reduction of a Linear Substitution to its Canonical Form: Prof. W. Burnside.

The following presents were made to the Library:—

Koenigsberger, L.—"The Investigations of Hermann von Helmholtz on the Fundamental Principles of Mathematics and Mechanics," 8vo; Washington, 1898 (from "Smithsonian Report," 1896, pp. 93–124).

Oltramare, G.—"Calcul de Généralisation," 8vo; Paris, 1899. Two copies: one presented by the Author and the other by the Publisher.

"Educational Times," January, 1899.

"Indian Engineering," Vol. xxiv., Nos. 21–25, Nov. 19–Dec. 17, 1898.

"Reciprocal Polygons," by Jamshedji Edalji, B.A., B.Sc.; Ahmedabad, 1898. From the Author.

The following is the list of exchanges received:—

"Proceedings of the Royal Society," Vol. lxiv., No. 405.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xxii., St. 11; Leipzig, 1898.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. xlii., Pt. 5, 1897–98.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig," Bd. l., Pt. 5, 1898.

"Proceedings of the Physical Society of London," Vol. xvi., Pt. 3; November, 1898.

"Proceedings of the Canadian Institute," Vol. i., Pt. 6; November, 1898.

"Proceedings of the Royal Irish Academy," Vol. v., No. 1; December, 1898.

"Bulletin of the American Mathematical Society," Series 2, Vol. v., No. 3; December, 1898.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Vol. iv., Fasc. 8–11; Napoli, 1898.

"Rendiconti del Circolo Matematico di Palermo," Tomo xii., Fasc. 6; November and December, 1898.

"Bulletin des S
"Acta Mathema
"Annali di Mate
"Atti della Reale
Fasc. 10, 11; Roma,

Zeroes of the Bess

Read January 12th,

In a previous paper, quantity, have been the results there obtai the real part of n is gre with the essential sin Stokes' formula. When $-m$, m being an inte associated with the esse can be derived from a previous paper. The $J_n(z)/z^n$ depended on the all points of the z-plane A solution of Bessel's e holomorphic function m it is so is that of n solutions of Bessel's equa solution Y_n, and that soluti positive imaginary infinity, the function K_n will be princ ... by K_n.

In §§ 1, 2, the elementary ... of the function K_n ... vestigated, and the function is ... to be that solution ... differential equation

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}-\left(1+\frac{n^2}{z^2}\right)y=0$$

whi ... positive infinity, the plane

by the negative imaginary axis, so that the variable z takes values from $re^{-\frac{1}{2}\pi\iota}$ to $re^{\frac{3}{2}\pi\iota}$, where $|z| = r$. An expression in the form of an integral is obtained for the product $K_n(a)\,K_n(b)$, where the real parts of a and b are positive, § 3; and hence an expression for $|K_n(z)|^2$, where the real part of z is positive. The zeroes associated with the essential singularity at infinity are shown to lie wholly at infinity, § 4. It is shown, in § 5, that $K_n(z)$ has no real zeroes unless $n = 2k+\frac{3}{2}$, k being an integer when it has one real negative zero; and, in § 6, it is shown that $K_n(z)$ has no pure imaginary zeroes. In § 7, it is shown that $K_n(z)$ has no zero whose real part is positive other than those at infinity. It is then shown, § 8, that, when $1 > n > 0, K_n(z)$ has no zeroes other than those at infinity; § 9, that, when $2 > n > 1$, it has one zero whose real part is negative; and, § 10, that, when $m+1 > n > m$, m being an integer, it has m zeroes whose real parts are negative. The case where n is an integer is discussed in § 11, where it is shown that $K_n(z)$ has n zeroes whose real parts are negative.

1. *The Function K_n.*

Writing ιz for z in Bessel's equation, it becomes

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}-\left(1+\frac{n^2}{z^2}\right)y=0,$$

and K_n becomes the solution of the new equation which vanishes at the real positive infinity. A solution of this equation is

$$A\int_{s_0}^{s_1} e^{\frac{1}{2}z(s+1/s)}\frac{ds}{s^{n+1}},$$

where the path of integration begins and ends at points where $e^{\frac{1}{2}z(s+1/s)}$ vanishes. The following values of s_0 and s_1 satisfy this condition:

$$s_0 = \infty\, e^{\alpha\iota}, \quad s_1 = \infty\, e^{\beta\iota},$$

$$s_0 = 0\, e^{-\alpha\iota}, \quad s_1 = 0\, e^{-\beta\iota},$$

$$s_0 = \infty\, e^{\iota}, \quad s_1 = 0\, e^{-\beta\iota},$$

where the real parts of $ze^{\alpha\iota}$, $ze^{\beta\iota}$ are less than zero.

Denoting these solutions by y_1, y_2, y_3 respectively,

$$y_1 = A\int_{\infty e^{\alpha\iota}}^{\infty e^{\beta\iota}} e^{\frac{1}{2}z(s+1/s)}\frac{ds}{s^{n+1}};$$

putting $sz = 2\sigma$,

$$y_1 = A\left(\frac{z}{2}\right)^n \int_{\infty e^{\alpha' \iota}}^{\infty e^{\beta' \iota}} e^{\sigma + z^2/4\sigma} \frac{d\sigma}{\sigma^{n+1}},$$

where the real parts of $e^{\alpha'\iota}$, $e^{\beta'\iota}$ are less than zero; that is,

$$y_1 = 2\pi\iota A\left(\frac{z}{2}\right)^n \sum_0^\infty \frac{\left(\frac{z}{2}\right)^{2r}}{\Pi(n+r)\,\Pi(r)},$$

and choosing A so that $2\pi\iota A = 1$,

$$y_1 = \left(\frac{z}{2}\right)^n \sum_0^\infty \frac{\left(\frac{z}{2}\right)^{2r}}{\Pi(n+r)\,\Pi(r)};$$

hence y_1 is the solution usually denoted by I_n.

Again, $$= \frac{1}{2\pi\iota} \int_{0e^{-\alpha\iota}}^{e^{-\beta\iota}} e^{\frac{1}{2}z(s+1/s)} \frac{ds}{s^{n+1}};$$

that is, writing $s\sigma = 1$,

$$y_2 = -\frac{1}{2\pi\iota} \int_{\infty e^{\alpha\iota}}^{\infty e^{\beta\iota}} e^{\frac{1}{2}z(\sigma+1/\sigma)} \frac{d\sigma}{\sigma^{-n+1}};$$

therefore $$y_2 = -I_{-n}.$$

To obtain the relation between y_1, y_2, and y_3, assume the real part of z to be positive and put $\alpha = \beta = \pi$; then

$$y_1 + y_2 = 2\iota \sin n\pi\, y_3,$$

that is, $$\int_0^\infty e^{-\frac{1}{2}z(s+1/s)} \frac{ds}{s^{n+1}} = \frac{\pi}{\sin n\pi}(I_{-n} - I_n).$$

The solution $\dfrac{\pi}{2\sin n\pi}(I_{-n} - I_n)$ will be denoted by K_n, then

$$K_n = \tfrac{1}{2} \int_0^\infty e^{-\frac{1}{2}z(s+1/s)} \frac{ds}{s^{n+1}} = \tfrac{1}{2} \int_0^\infty e^{-\frac{1}{2}z(s+1/s)} s^{n-1}\, ds.$$

The same result follows, when the real part of z is negative, by putting $\alpha = \beta = 0$. Using the result obtained (*Proceedings*, Vol. XXIX., p. 115), K_n is also given by

$$K_n = \frac{e^{-z}}{\Pi(n-\frac{1}{2})} \sqrt{\frac{\pi}{2z}} \int_0^\infty e^{-s} s^{n-\frac{1}{2}} \left(1 + \frac{s}{2z}\right)^{n-\frac{1}{2}} ds,$$

where the real part of n is greater than $-\frac{1}{2}$. Similarly, the solution which vanishes at the real negative infinity is

$$\frac{e^z}{\Pi(n-\frac{1}{2})}\sqrt{\frac{\pi}{2z}}\int_0^\infty e^{-s}s^{n-\frac{1}{2}}\left(1-\frac{s}{2z}\right)^{n-\frac{1}{2}}ds.$$

That these solutions may be single valued, it is necessary to bound the z plane by a straight line drawn from the origin to infinity; it is convenient to choose for this purpose the negative imaginary axis. The relation

$$K_n = \frac{\pi}{2\sin n\pi}(I_{-n}-I_n)$$

gives the means of expressing K_n when z is in the second or third quadrant in terms of functions with the variable in the first or fourth quadrant. From the above,

$$K_n(ze^{\pi\iota}) = \frac{\pi}{2\sin n\pi}\{e^{-n\pi\iota}I_{-n}(z)-e^{n\pi\iota}I_n(z)\};$$

that is,
$$K_n(ze^{\pi\iota}) = e^{-n\pi\iota}K_n(z)-\pi\iota I_n(z).$$

2. *Recurrence Formulæ and Difference Equation.*

In the equation

$$K_n = \tfrac{1}{2}\int_0^\infty e^{-\frac{1}{2}z(s+1/s)}\frac{ds}{s^{n+1}}$$

put $sz = 2\sigma$; then

$$2K_n = \left(\frac{z}{2}\right)^n\int_0^\infty e^{-\sigma-z^2/4\sigma}\frac{d\sigma}{\sigma^{n+1}};$$

hence, differentiating,

$$\frac{d}{dz}\left(\frac{K_n}{z^n}\right) = -\frac{K_{n+1}}{z^n},$$

or

$$\frac{dK_n}{dz} = \frac{n}{z}K - K_{n+1}.$$

Again, putting $2s = z\sigma$,

$$2K_n = \left(\frac{2}{z}\right)^n\int_0^\infty e^{-1/\sigma-\frac{1}{4}z^2\sigma}\frac{d\sigma}{\sigma^{n+1}}$$

and, differentiating,
$$\frac{d}{dz}(z^nK_n) = -z^nK_{n-1},$$

or

$$\frac{dK_n}{dz}+\frac{n}{z}K_n = -K_{n-1}.$$

Eliminating $\dfrac{dK_n}{dz}$, the difference equation is

$$K_{n+1}-\frac{2n}{z}K_n-K_{n-1}=0.$$

3. *Expression of $K_n(a)\,K_n(b)$ as an Integral.*

In the relation

$$z^n K_n(z)=\tfrac{1}{2}\int_0^\infty e^{-\frac{1}{2}(s+z^2/s)}\,s^{n-1}\,ds$$

put
$$z=\sqrt{u^2-v^2};$$

then
$$(\sqrt{u^2-v^2})^n K_n(\sqrt{u^2-v^2})=\tfrac{1}{2}\int_0^\infty e^{-\frac{1}{2}(s+u^2/s-v^2/s)}\,s^{n-1}\,ds.$$

Now
$$\int_{-\infty}^\infty e^{-\frac{1}{2}sy^2-vy}\,dy=\sqrt{\frac{2\pi}{s}}\,e^{v^2/2s};$$

therefore

$$\{\sqrt{(u^2-v^2)}\}^n K_n\{\sqrt{(u^2-v^2)}\}$$
$$=\frac{1}{2\sqrt{(2\pi)}}\int_0^\infty\int_{-\infty}^\infty e^{-\frac{1}{2}(s+sy^2+u^2/s)-vy}\,s^{n-\frac{1}{2}}\,dy\,ds;$$

that is, putting $s(1+y^2)=\sigma$,

$$\{\sqrt{(u^2-v^2)}\}^n K_n\{\sqrt{(u^2-v^2)}\}$$
$$=\frac{1}{2\sqrt{(2\pi)}}\int_0^\infty\int_{-\infty}^\infty e^{-\frac{1}{2}[\sigma+u^2(1+y^2)/\sigma]-vy}\,\frac{\sigma^{n-\frac{1}{2}}}{(1+y^2)^{n+\frac{1}{2}}}\,dy\,d\sigma;$$

whence, changing the order of integration, and integrating with respect to σ,

$$\{\sqrt{(u^2-v^2)}\}^n K_n\{\sqrt{(u^2-v^2)}\}$$
$$=\frac{u^{n+\frac{1}{2}}}{\sqrt{(2\pi)}}\int_{-\infty}^\infty e^{-vy}K_{n+\frac{1}{2}}\{u\sqrt{(1+y^2)}\}\,\frac{dy}{(1+y^2)^{\frac{1}{2}(2n+1)}},\tag{A}$$

where, the path of integration being along the real axis, the rea parts of $u\pm v$ are positive. When this condition is not satisfied, the path of integration of the integral on the right-hand side must be such that the integral is convergent. Writing

$$y=\sinh\phi,\quad u=a\cosh\chi,\quad v=a\sinh\chi+\iota b,$$

the relation becomes

$$\{\sqrt{(a^2+b^2-2\iota ab\sinh\chi)}\}^n K_n\{\sqrt{(a^2+b^2-2\iota ab\sinh\chi)}\}$$

$$=\frac{(a\cosh\chi)^{n+\frac{1}{2}}}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}e^{-\iota b\sinh\chi-a\sinh\phi\sinh\chi}K_{n+\frac{1}{2}}(a\cosh\phi\cosh\chi)\frac{d\phi}{(\cosh\phi)^{n-\frac{1}{2}}}.$$

Divide both sides by $\cosh^{2n}\chi$, and integrate between $-\infty$ and $+\infty$; then

$$W=\int_{-\infty}^{\infty}\{\sqrt{(a^2+b^2-2\iota ab\sinh\chi)}\}K_n\{\sqrt{(a^2+b^2-2\iota ab\sinh\chi)}\}\frac{d\chi}{\cosh^{2n}\chi}$$

$$=\frac{a^{n+\frac{1}{2}}}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\iota b\sinh\chi-a\sinh\phi\sinh\chi}\frac{K_{n+\frac{1}{2}}(a\cosh\phi\cosh\chi)\,d\chi\,d\phi}{\cosh^{n-\frac{1}{2}}\phi\cosh^{n-\frac{1}{2}}\chi};$$

that is, by the above,

$$W=a^{n+\frac{1}{2}}\int_{-\infty}^{\infty}e^{-\iota b\sinh\chi}\frac{a^nK_n(a)\,d\phi}{(a\cosh\phi)^{n+\frac{1}{2}}\cosh^{n-\frac{1}{2}}\phi};$$

hence $$W=2a^nK_n(a)\int_0^{\infty}\frac{\cos(b\sinh\phi)}{\cosh^{2n}\phi}\,d\phi;$$

that is, $$W=2a^nK_n(a)\int^{\infty}\frac{\cos by}{+y^2)^{n+\frac{1}{2}}}\,dy.$$

Now $$2z^nK_n(z)=\int_0^{\infty}e^{-\frac{1}{2}(s+z^2/s)}s^{n-1}\,ds;$$

that is, $$2z^nK_n(z)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\int_0^{\infty}e^{\frac{1}{2}s(1+y^2)}s^{n-\frac{1}{2}}\cos zy\,ds\,dy;$$

hence $$2z^nK_n(z)=2^{n+1}\Pi(n-\tfrac{1}{2})\int_0^{\infty}\frac{\cos zy}{(1+y^2)^{n+\frac{1}{2}}}\,dy;$$

therefore $$W=\frac{\sqrt{\pi}}{2^{n-1}\Pi(n-\frac{1}{2})}a^nb^nK_n(a)K_n(b);$$

and hence

$$K_n(a)K_n(b)=\frac{2^{n-1}\Pi(n-\frac{1}{2})}{\sqrt{\pi}\,a^nb^n}\int_{-\infty}^{\infty}\{\sqrt{(a^2+b^2-2\iota ab\sinh\chi)}\}^n$$

$$\times K_n\{\sqrt{(a^2+b^2-2\iota ab\sinh\chi)}\}\frac{d\chi}{\cosh^{2n}\chi}.$$

From this it follows that

$$K_n(a)K_n(b)=\frac{2^{n-2}\Pi(n-\frac{1}{2})}{\sqrt{\pi}\,a^nb^n}\int_{-\infty}^{\infty}\int_0^{\infty}e^{-\frac{1}{2}[s+(a^2+b^2-2\iota ab\sinh\chi)/s]}\frac{s^{n-1}}{\cosh^{2n}\chi}\,ds\,d\chi,$$

that is, $$K_n(a)K_n(b)=\tfrac{1}{2}\int_0^{\infty}e^{-\frac{1}{2}[s+(a^2+b^2)/s]}K_n\left(\frac{ab}{s}\right)\frac{ds}{s};$$

which holds for all values of a and b whose real parts are positive. This at once gives an expression for $| K_n |^2$; putting

$$a = re^{\phi}, \quad b = re^{-\phi\iota},$$

$$| K_n |^2 = \tfrac{1}{2} \int_0^\infty e^{-\frac{1}{2}s - (r^2 \cos 2\phi)/s} K_n \left(\frac{r^2}{s}\right) \frac{ds}{s}.$$

4. *The Zeroes of the Function $K_n(z)$ which are associated with the essential singularity at Infinity all have for real part $+\infty$.*

This follows immediately from the expression for $K_n(z)$ in the neighbourhood of infinity, which is

$$\sqrt{\left(\frac{\pi}{2z}\right)} e^{-z} \sum_0^\infty \frac{\Pi(n+r-\frac{1}{2})}{\Pi(n-r-\frac{1}{2})\,\Pi(r)} \frac{1}{(2z)^r},$$

the zeroes associated with the essential singularity being those of e^{-z}.

5. *Real Zeroes of $K_n(z)$ other than that at Infinity.*

Such a zero must be a zero of the function

$$\frac{1}{\Pi(n-\frac{1}{2})} \int_0^\infty e^{-s} s^{n-\frac{1}{2}} \left(1+\frac{s}{2z}\right)^{n-\frac{1}{2}} ds.$$

Now, when z is real and positive, this expression is positive and greater than unity; therefore $K_n(z)$ has no real positive zero other than that at infinity. When z is real and negative,

$$K_n(z) = e^{-n\pi\iota} K_n(z') - \pi\iota I_n(z'),$$

where $z' = -z$, and is real and positive; hence, that $K_n(z)$ should have a real negative zero, it is necessary that $e^{-n\pi\iota} K_n(z') - \pi\iota I_n(z')$ should vanish for a real positive value of z'. Now, $K_n(z')$ and $I_n(z')$ being two independent solutions of a linear differential equation of the second order, they cannot vanish together; and therefore, since $K_n(z')$ and $I_n(z')$ are both real, $K_n(z)$ has no real negative zero unless $e^{-n\pi\iota}$ is a pure imaginary. This requires $n = p+\frac{1}{2}$, where p is an integer, and then z' has to satisfy the equation

$$\pi I_{p+\frac{1}{2}}(z') + (-)^p K_{p+\frac{1}{2}}(z') = 0.$$

The function $I_{p+\frac{1}{2}}(z')$ is positive for all real positive values of z', and increases continuously from zero to infinity as z' increases from zero to infinity; the function $K_{p+\frac{1}{2}}(z')$ is positive for all real positive values of z', and decreases continuously from infinity to zero as z' increases from zero to infinity. That a positive real value of z' may satisfy the

above equation, p must be an odd integer, and then there is only one value of z' satisfying it, for $K_{p+\frac{1}{2}}(z')$ and $I_{p+\frac{1}{2}}(z')$ can only once be equal as z' goes from zero to infinity. Hence $K_n(z)$ has no real zero other than that at infinity unless $n = 2k+\frac{3}{2}$, where k is an integer, and it then has one real negative zero. In this case

$$K_n(z) = \frac{e^{-z}}{z^{2k+\frac{3}{2}}} X_{2k+1},$$

where X_{2k+1} is a rational integral algebraic expression of degree $2k+1$, which clearly has one real root.

6. *Pure Imaginary Zeroes of $K_n(z)$.*

When z is a pure imaginary $\pm \iota y$, y being real, the equation

$$K_n(\pm \iota y) = 0$$

is equivalent to

$$J_{-n}(y) = e^{\pm n\pi\iota} J_n(y)$$

when n is not an integer, and to

$$Y_n(y) = \pm \pi\iota J_n(y)$$

when n is an integer. Now, when n is not an integer, $e^{\pm n\pi\iota}$ is a complex number, and the first of the above equations cannot be satisfied unless $J_n(y)$ and $J_{-n}(y)$ vanish for the same value of y; which is impossible, as they are independent solutions of the differential equation. When n is an integer, the second equation cannot be satisfied unless $Y_n(y)$ and $J_n(y)$ vanish together; which is impossible, for the same reason. Hence $K_n(z)$ has no pure imaginary zeroes.

7. *The Function $K_n(z)$ has no Zeroes in the First or Fourth Quadrant other than those at Infinity.*

Let

$$z = re^{\phi\iota},$$

where

$$\frac{\pi}{2} > \varphi > 0;$$

then

$$|K_n(z)|^2 = \tfrac{1}{2}\int_0^\infty e^{-\frac{1}{2}s-(r^2\cos 2\phi)/s} K_n\left(\frac{r^2}{s}\right)\frac{ds}{s}.$$

Now, each element of the integral on the right-hand side is positive, and cannot vanish if r is finite; hence $|K_n(z)|^2$ cannot vanish for a value of z in the first or fourth quadrant. Therefore $K_n(z)$ has no zeroes *in the first or fourth* quadrant other than those at infinity.

8. *When* $1 > n > 0$, $K_n(z)$ *has no Zeroes other than those at Infinity.*

From the relation

$$K_n(z) = \frac{\pi}{2\sin n\pi}\{I_{-n}(z) - I_n(z)\},$$

it follows that the zeroes of $K_n(z)$ are identical with those of $I_{-n}(z) - I_n(z)$. Writing

$$I_n(z) = z^n u_n,$$

the zeroes of $K_n(z)$ are the same as those of

$$z^{2n} u_n - u_{-n} = 0;$$

and therefore, when they exist, are continuous with those u_n, u_{-n}, and z^{2n}. Now all the zeroes of u_n and u_{-n}, when $1 > n > 0$, are associated with the essential singularity at infinity; and therefore, if, when $1 > n > 0$, $K_n(z)$ has zeroes other than those at infinity, they must be continuous with the zeroes of z^{2n}. Now, $f(z)$ and $\phi(z)$ being holomorphic functions, the zeroes of $z^n f(z) - \phi(z)$ continuous with $z = 0$ are included among the zeroes of

$$z = \kappa\left\{\frac{\phi(z)}{f(z)}\right\}^{1/n},$$

where $$|\kappa| = 1.$$

That this latter equation should have zeroes continuous with $z = 0$, it is necessary that it should be possible to draw a contour surrounding $z = 0$ for each point of which

$$\left|\left\{\frac{\phi(z)}{z^n f(z)}\right\}^{1/n}\right| < 1,$$

or, what is the same thing, a contour for each point of which

$$\left|\frac{\phi(z)}{z^n f(z)}\right| < 1.$$

Hence, that $K_n(z)$ should have a zero continuous with $z = 0$, it is necessary that it should be possible to draw a contour enclosing the origin for each point of which

$$\left|\frac{I_{-n}(z)}{I_n(z)}\right| < 1.$$

Now, when $1 > n > 0$, $\sin n\pi$ is positive; therefore $I_{-n}(z) > I_n(z)$ for

all real positive values of z, for $K_n(z)$ is positive for all real positive values of z, § 5; therefore no such contour can be drawn, and hence, when $1 > n > 0$, $K_n(z)$ has no zeroes other than those at infinity.

9. *When* $2 > n > 1$, $K_n(z)$ *has one Zero whose real part is negative.*

Let the series of contours $|z^n K_n(z)| = \text{constant}$ be drawn; then, for the portions of these contours on the right-hand side of the imaginary axis,

$$|z^n K_n(z)|^2 = \frac{r^{2n}}{2}\int_0^\infty e^{-\frac{1}{2}s-(r^2\cos 2\phi)/s} K_n\left(\frac{r^2}{s}\right)\frac{ds}{s},$$

where $z = re^{\phi\iota}$; keeping r constant, each element of the integral increases as ϕ increases from 0 to $\frac{\pi}{2}$, and therefore the integral increases. Now, $n > 0$, when $\phi = 0$, $|z^n K_n(z)|$ decreases from $2^{n-1}\Pi(n-1)$ to zero as r increases from zero to infinity; therefore $|z^n K_n(z)|$ takes the value of $r_0^n K_n(r_0)$ once in the first quadrant for a value of z such that $|z| = r$, r being any quantity greater than r_0, and also once in the fourth quadrant, the two points where $|z^n K_n(z)|$ takes this value, when $|z| = r$, being equidistant from the real axis. Hence, $n > 0$, the portions of the contours

$$|z^n K_n(z)| = r_0^n K_n(r_0)$$

on the right-hand side of the imaginary axis are symmetric with respect to the real axis, of parabolic form, and enclose all the zeroes associated with the essential singularity at infinity. When $1 > n > 0$, the contours

$$|z^n K_n(z)| = r_0^n K_n(r_0)$$

lie wholly on the right-hand side of the imaginary axis; for, if there are other portions of them on the left-hand side, these portions must be parabolic or closed curves. They cannot be parabolic, for $K_n(z)$ has no zeroes whose real part is $-\infty$, and $r_0^n K_n(r_0)$ cannot exceed $2^{n-1}\Pi(n-1)$; and they cannot be closed curves, for then $K_n(z)$ would have zeroes other than those at infinity, which is contrary to § 8. When $2 > n > 1$, the portion of the contour

$$z^n K_n(z) = r^n K_n(r_0)$$

* The contours $|z^n K_n(z)| = r_0^n K_n(r_0)$ do not meet the imaginary axis, except when $r_0 = 0$; and therefore $z^n K_n(z)$ is a holomorphic function throughout the spaces enclosed by these contours.

on the left of the imaginary axis will consist of one closed curve. There can be no parabolic curve belonging to this contour on the left of the imaginary axis, for the same reason as above; hence, if there is any portion of the contour to the left of the imaginary axis, it must consist of one or more closed curves. There cannot be more than one closed curve, for, if there were, $\frac{d}{dz}\{z^n K_n(z)\}$ would have more than one zero to the left of the portion of the contour

$$| z^n K_n(z) | = r^n K_n(r_0),$$

which lies on the right of the imaginary axis. Now

$$\frac{d}{dz}\{z^n K_n(z)\} = -z^n K_{n-1}(z),$$

which, when $2 > n > 1$, possesses only one such zero, viz., $z = 0$; hence there is at most one closed curve on the left of the imaginary axis forming a portion of the contour

$$| z^n K_n(z) | = r_0^n K_n(r_0).$$

To establish the existence of this closed curve, it is sufficient to show that, for values of r_0 lying between zero and a certain value, there are two points on the negative real axis where

$$| z^n K_n(z) | = r_0^n K_n(r_0).$$

When $z = re^{\pi\iota}$,

$$z^n K_n(z) = r^n K_n(r) - \pi\iota e^{n\pi\iota} r^n I_n(r);$$

that is,

$$z^n K_n(z) = \lambda - \iota\mu,$$

where

$$\lambda = r^n \{K_n(r) + \pi \sin n\pi\, I_n(r)\},$$

and

$$\mu = \pi \cos n\pi\, r^n I_n(r);$$

hence

$$| z^n K_n(z) |^2 = \lambda^2 + \mu^2,$$

and

$$\frac{d}{dr}\{| z^n K_n(z) |^2\} = 2\left(\lambda \frac{d\lambda}{dr} + \mu \frac{d\mu}{dr}\right),$$

where

$$\frac{d\lambda}{dr} = -r^n \{K_{n-1}(r) - \pi \sin n\pi\, I_n(r)\},$$

$$\frac{d\mu}{dr} = \pi \cos n\pi\, r^n I_{n-1}(r).$$

Now $I_n(r)$, $I_{n-1}(r)$ increase with r, when $2 > n > 1$; hence $\mu \frac{d\mu}{dr}$ is positive, and increases from zero to infinity as r increases

from zero to infinity, except when $n=\frac{3}{2}$, in which case it is always zero. Again, $2>n>1$, λ is positive when $r=0$, for it is then equal to $2^{n-1}\Pi(n-1)$, and diminishes as r increases; $\frac{d\lambda}{dr}$ is negative for all values of r, and diminishes at first as r increases; therefore $\lambda\frac{d\lambda}{dr}$ is negative for values of r less than that for which λ vanishes, and diminishes to zero as r increases to this value. Hence $\lambda\frac{d\lambda}{dr}+\mu\frac{d\mu}{dr}$ vanishes for some value r_1 of r, where $r_1<\rho$, and

$$K_n(\rho)+\pi\sin n\pi I_n(\rho)=0,^*$$

when n is different from $\frac{3}{2}$; and, when $n=\frac{3}{2}$, $-\rho$ is a zero of $K_{\frac{3}{2}}(z)$. Therefore $\lambda^2+\mu^2$ diminishes to a minimum as r increases from zero to r_1, and then increases as r increases from r_1 to infinity; that is, there are two points on the negative real axis where

$$|\,z^nK_n(z)\,|=r_0^nK_n(r_0)$$

for values of r_0 less than that for which

$$r_0^nK_n(r_0)=r_1^n\,|\,K_n(r_1e^{\pi i})\,|\,.$$

Hence, when $2>n>1$, the contour

$$|\,z^nK_n(z)\,|=r_0^nK_n(r_0)$$

consists of a parabolic portion to the right of the imaginary axis and a closed curve to the left of the imaginary axis; that is, $K_n(z)$ has, in addition to the zeroes at infinity, one zero of finite modulus whose real part is negative.

This result can also be obtained as follows:—The function $K_{n-1}(z)/z^{n-1}$, when $2>n>1$, has no zeroes other than those at infinity, has a pole at the origin ($z=0$), and is infinite at the negative real infinity. Imagining the contours $\left|\frac{K_{n-1}(z)}{z^{n-1}}\right|$ constant to be drawn, they consist, for great values of the constant, of a parabolic portion to the left of the imaginary axis and a curve round the origin starting from a point on the negative imaginary axis enclosing the origin and ending at a point on the negative imaginary axis nearer the origin than the point it started from; as the constant diminishes,

* There is a real value of ρ satisfying $K_n(\rho)+\pi\sin n\pi I_n(\rho)=0$, for $\sin n\pi$ is negative, $K_n(\rho)$ diminishes from infinity to zero as ρ increases from zero to infinity, and $I_n(\rho)$ increases from zero to infinity as ρ increases.

these two curves approach until they coalesce at a double point, after which the curve ultimately comes to be wholly in the first and fourth quadrant enclosing the zeroes at infinity. This double point is a point where $\frac{d}{dz}\left(\frac{K_{n-1}(z)}{z^{n-1}}\right)$ vanishes; that is, it is a zero of $K_n(z)$, and the only one at a finite distance from the origin, as the curves cannot coalesce in more than one double point.

10. *When $m+1>n>m$, m being a positive integer, $K_n(z)$ has, in addition to the zeroes at infinity, m zeroes whose real parts are negative.*

Imagining the contours $|\, z^n K_n(z)\,|$ constant to be drawn, they consist of parabolic portions to the right of the imaginary axis and one or more closed curves to the left, so long as the constant is less than $2^{n-1}\Pi(n-1)$; hence $z^n K_n(z)$ cannot have more than one more zero to the left of the imaginary axis than $z^{n-1}K_{n-1}(z)$, for

$$\frac{d}{dz}\{z^n K_n(z)\} = -z^n K_{n-1}(z),$$

the zero $z=0$ of $z^n K_{n-1}(z)$ separates the zeroes of $z^n K_n(z)$ at infinity from those to the left of the imaginary axis, so that for any value of $|\, z^n K_n(z)\,|$ less than $2^{n-1}\Pi(n-1)$ there cannot be more than one closed curve more than the number of zeroes of $z^{n-1}K_{n-1}(z)$ to the left of the imaginary axis. Therefore the number of zeroes of $K_n(z)$ to the left of the imaginary axis cannot exceed m, for, § 9, when $m=1$, $z^n K_n(z)$ has only one such zero. Again,

$$\frac{d}{dz}\left\{\frac{K_{n-1}(z)}{z^{n-1}}\right\} = -\frac{K_n(z)}{z^{n-1}};$$

therefore, imagining the contours $\left|\frac{K_{n-1}(z)}{z^{n-1}}\right|$ constant to be drawn, they consist, for large values of the constant, of a parabolic portion to the left of the imaginary axis and a curve, as in § 9, round the origin; as the constant diminishes these curves approach each other, and, ultimately, after passing through double points, consist of a parabolic portion to the right of the imaginary axis and closed curves surrounding the zeroes of $K_{n-1}(z)$; hence one more double point has to be passed through than there are zeroes of $K_{n-1}(z)$ to the left of the imaginary axis; that is, $K_n(z)$ has one more zero to the left of the imaginary axis than $K_{n-1}(z)$. Therefore $K_n(z)$ has m zeroes to the left of the imaginary axis.

11. *When n is an integer, $K_n(z)$ has n zeroes to the left of the imaginary axis.*

The function $K_n(z)$, considered as a function of n, possesses singularities corresponding to each integral value of n; hence, when n passes through an integral value, zeroes of $K_n(z)$, considered as a function of z, may either appear or disappear. Now, $K_n(z)$ is the same function as $K_{-n}(z)$; therefore, when n passes through the value zero, it may be inferred that no new zero appears and that none disappear, as $K_n(z)$ possesses the same zeroes as $K_{-n}(z)$. This can also be shown in the following way :—The zeroes of $K_0(z)$ are those of $\frac{dI_{-n}}{dn}-\frac{dI_n}{dn}$, where $n=0$; that is, of

$$\frac{du_{-n}}{dn}-\frac{du_n}{dn}+(u_{-n}+u_n)\log z,$$

where $$z^n u_n = I_n \quad \text{and} \quad n=0.$$

Now, this expression has its zeroes continuous with those of $\frac{du_{-n}}{dn}-\frac{du_n}{dn}$, $u_{-n}+u_n$, and $\log z$, and all the zeroes of $\frac{du_{-n}}{dn}-\frac{du_n}{dn}$ and $u_{-n}+u_n$ are associated with the essential singularity at infinity; therefore, if $K_0(z)$ has a zero other than those at infinity, it is continuous with the zero of $\log z$. The function $K_0(z)$ may be written

$$I_0(z)(\log z+\gamma-\log 2)-f(z^2),$$

from which it appears that the zero continuous with that of $\log z$ would be real if it existed; hence there is no such zero, and therefore $K_0(z)$ has no zero other than those at infinity. It can be shown, exactly as in § 9, that the contour

$$|\, zK_1(z)\, | = r_0 K_1(r_0)$$

cuts the negative real axis in two points, provided r_0 does not exceed a certain value; and therefore there is at least one zero of $K_1(z)$ to the left of the imaginary axis. There cannot be more than one zero of $K_1(z)$ to the left of the imaginary axis, for $K_0(z)$ has no zero other than at infinity; therefore $K_1(z)$ has, in addition to the zeroes at infinity, one zero whose real part is negative. Otherwise, imagining the contours $|\, K_0(z)\, |$ constant drawn; they consist, for large values of the constant, of a curve round the origin, as in § 9, and a parabolic portion to the left of the imagining axis at a great distance from it; when the constant has been sufficiently diminished, the contour con-

sists of a parabolic curve to the right of the imaginary axis enclosing the zeroes of $K_0(z)$ at infinity; hence, for some value of the constant, the parabolic portion to the left of the imaginary axis and the curve round the origin must coalesce into a single curve, the point at which they coalesce being a double point on the contour. This point is a zero of $K_1(z)$, for, at a double point,

$$\frac{dK_0(z)}{dz} = 0,$$

that is, $$K_1(z) = 0;$$

hence as before, $K_1(z)$ has one zero, whose real part is negative, as the curves cannot coalesce in more than one point. Proceeding as in §10, it follows that $K_n(z)$ has n zeroes whose real parts are negative.

These zeroes are associated with the singularity at the origin, but cannot be approximated to in the same manner as the corresponding zeroes of I_{-n}, for the function cannot be expanded in a power series in the neighbourhood of the origin, and therefore Lagrange's expansion theorem does not apply. When n is not an integer, they could be obtained from the corresponding zeroes of I_{-n} by the methods of the previous paper, but the calculation would be extremely laborious. Another method would be to approximate to them from the zeroes of $K_{m+\frac{1}{2}}(z)$, where m is the integer next least to n. Except in the case when n is half an odd integer, there are other zeroes of $K_n(z)$, in addition to those found above, associated with the singularity at the origin, but they correspond to values $re^{i\theta}$ of z when θ does not lie between $-\frac{\pi}{2}$ and $\frac{3\pi}{2}$. The zeroes of any solution of the equation can be found, for, if $Z_n(z)$ is that solution which vanishes at the real negative infinity, its zeroes are the images of those of $K_n(z)$ in the imaginary axis (the plane being bounded as before), and any solution is a linear function of $K_n(z)$ and $Z_n(z)$; therefore its zeroes can be found. It will have, at most, $2m$ zeroes not associated with the essential singularity at infinity, the plane being bounded as above.

On the Reduction of a Linear Substitution to its Canonical Form. *By* W. BURNSIDE. Received January 9th, 1899. Read January 12th, 1899. Received, in revised form, March 1st, 1899.

Introduction.

A canonical form, to which any linear substitution of non-vanishing determinant may be reduced, was first given by M. Jordan in his *Traité des Substitutions &c.* (pp. 114–126). M. Jordan's investigation applies directly to linear substitutions of the form

$$x'_s \equiv a_{s1}x_1 + a_{s2}x_2 + \ldots + a_{sn}x_n, \quad (\text{mod. } p),$$
$$(s = 1, 2, \ldots, n),$$

where p is a prime; but the canonical form at which he arrives is known to hold equally when the congruences are replaced by equations.

The canonical form may be written as follows:—

$$\begin{array}{llll}
y'_1 = \lambda y_1, & y'_r = \lambda y_r + y_{r-1}, & (r = 2, 3, \ldots, a_1), \\
y'_{a_1+1} = \lambda y_{a_1+1}, & y'_{a_1+r} = \lambda y_{a_1+r} + y_{a_1+r-1}, & (r = 2, 3, \ldots, a_2), \\
y'_{a_1+a_2+1} = \lambda y_{a_1+a_2+1}, & y'_{a_1+a_2+r} = \lambda y_{a_1+a_2+r} + y_{a_1+a_2+r-1}, & (r = 2, 3, \ldots, a_3), \\
\ldots & \ldots & \ldots \\
z'_1 = \mu z_1, & z'_r = \mu z_r + z_{r-1}, & (r = 2, 3, \ldots, b_1), \\
z'_{b_1+1} = \mu z_{b_1+1}, & z'_{b_1+r} = \mu z_{b_1+r} + z_{b_1+r-1}, & (r = 2, 3, \ldots, b_2), \\
\ldots & \ldots & \ldots \\
w'_1 = \nu w_1, & w'_r = \nu w_r + w_{r-1}, & (r = 2, 3, \ldots, c_1). \\
\ldots & \ldots & \ldots
\end{array}$$

The numbers $a_1, a_2, \ldots, b_1, \ldots, c_1, \ldots$ are such that

$$a_1 + a_2 + a_3 + \ldots + b_1 + b_2 + \ldots + c_1 + \ldots + \ldots = n,$$

where n is the number of independent variables affected by the substitution. It will be assumed in what follows that the a's, b's, ..., are written so that

$$\begin{array}{l}
a_1 \geqslant a_2 \geqslant a_3 \geqslant \ldots\ldots, \\
b_1 \geqslant b_2 \geqslant \ldots\ldots\ldots\ldots, \\
c_1 \geqslant \ldots\ldots\ldots\ldots\ldots\ldots. \\
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots
\end{array}$$

The quantities λ, μ, ν, ... are the *distinct* roots of the characteristic equation

$$\begin{vmatrix} a_{11}-\theta, & a_{12}, & a_{13}, & \dots, & a_{1n} \\ a_{21}, & a_{22}-\theta, & \dots\dots\dots, & & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1}, & a_{n2}, & \dots\dots\dots, & & a_{nn}-\theta \end{vmatrix} = 0.$$

The y's, z's, ... are independent linear functions of the original variables with coefficients depending on λ, μ, ..., respectively, and on the coefficients of the given substitution.

The necessary and sufficient conditions that two linear substitutions A and B in the same number of variables should be capable of being transformed the one into the other, *i,e.*, that it should be possible to find a third linear substitution C such that

$$C^{-1}AC = B,$$

were first given by Weierstrass in his memoir "Zur Theorie der bilinearen und quadratischen Formen" (*Berliner Monatsberichte*, 1868, and *Mathematische Werke*, Vol. II., pp. 19–44). The conditions are that, for each value of r from 0 to $n-1$, the greatest common measure of all the r^{th} minors of the characteristic determinant of A should be equal to the greatest common measure of all the r^{th} minors of that of B.

Two substitutions which can be transformed the one into the other clearly can be reduced to the same canonical form; and the invariant elements of either, namely, the greatest common measure of all the r^{th} minors of the characteristic determinant ($r = 0, 1, \dots, n-1$) must be expressible in terms of the various constants which enter into the canonical form. This identification has been carried out by Netto, "Zur Theorie der linearen Substitutionen" (*Acta Mathematica*, Vol. XVII., pp. 267–271). He shows in fact that, the characteristic determinant of the canonical form (p. 180) being

$$(\lambda-\theta)^{a_1+a_2+a_3+\dots}(\mu-\theta)^{b_1+b_2+b_3+\dots}(\nu-\theta)^{c_1+c_2+c_3+\dots}\dots,$$

the greatest common measure of its first minors is

$$(\lambda-\theta)^{a_2+a_3+\dots}(\mu-\theta)^{b_2+b_3+\dots}(\nu-\theta)^{c_2+c_3+\dots}\dots,$$

that of its second minors

$$(\lambda-\theta)^{a_3+\dots}(\mu-\theta)^{b_3+\dots}(\nu-\theta)^{c_3+\dots}\dots,$$

and so on.

It is clear that, for the actual reduction of a substitution to its canonical form, the roots $\lambda, \mu, \nu, \ldots$ of the characteristic equation must be known; in other words, the operations necessary must be carried out in a region of rationality which contains $\lambda, \mu, \nu, \ldots$ as well as the coefficients of the substitution. In the general case, *i e.*, when all the roots of the characteristic equation are different, the operation presents no difficulty. If this case and that in which the substitution is of finite order are put on one side, no general method has hitherto been given, so far as I know, for effecting the reduction.

A method always effective for this purpose might be regarded from two points of view. Formally, it is equivalent to an independent proof of the accuracy of the canonical form; and, thence, on the lines of Herr Netto's determination just quoted, *a posteriori* to a verification of Weierstrass's conditions of equivalence for two substitutions. From this point of view it is quite immaterial whether the characteristic equation can be solved arithmetically or not. On the other hand, if the roots of the characteristic equation can be actually obtained by the four rules of arithmetic and the extraction of roots, such a method is an actual and feasible process of calculation by which the canonical form can be arrived at. It is the object of the present paper to give such a method. The leading idea is to consider the effect of the substitution on a linear function of the n variables with n arbitrary coefficients. This gives rise to a new substitution on a number of variables which cannot be greater than n, but may be less. The reduction of this new substitution to its canonical form can be carried out by a process which is the same in all cases; and on the reduction of this substitution that of the original substitution is made to depend.

It may be pointed out, as arising from the final result of the reduction, that the characteristic determinant of the new substitution spoken of is the result of dividing the characteristic determinant of the original substitution by the greatest common measure of its first minors; *i.e.*, it is the first "invariant factor" (*Elementar-theiler*) of the characteristic determinant of the original substitution. This is the same as saying that the characteristic equation of the new substitution is the equation of lowest degree which the matrix of the original substitution satisfies.*

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* This Introduction has been recast at the suggestion of one of the referees; no *other* alteration has been made.

1. Let
$$x'_s = \Sigma a_{st} x_t, \quad (s, t = 1, 2, \ldots, n),$$
be a linear substitution S, on n variables, whose determinant is not zero.

Let
$$y_1 = \sum_1^n A_r x_r$$
be a homogeneous linear function of the variables with arbitrary coefficients; and let
$$y_1, y_2, y_3, \ldots$$
be a series of homogeneous linear functions of the variables, each of which is derived from the preceding one by carrying out the substitution S on the variables. Since there are only n variables, not more than n y's can be linearly independent; but it may happen that, though the coefficients in y_1 are arbitrary, the number of linearly independent y's is less than n.

Suppose that y_{m+1} is the first of the y's that can be expressed linearly in terms of those that precede it, and that
$$y_{m+1} + a_m y_1 + a_{m-1} y_2 + \ldots + a_1 y_m = 0.$$
The constants $a_1, a_2, \ldots, a_m$ are functions of the coefficients of the substitution S; and every one of the y's can be expressed linearly in terms of the first m.

So far as it affects the y's, the substitution S takes the form
$$\left.\begin{aligned} y'_1 &= y_2, \\ y'_2 &= y_3, \\ \ldots &\ \ldots \\ y'_{m-1} &= y_m, \\ y'_m &= -a_m y_1 - a_{m-1} y_2 - \ldots - a_1 y_m \end{aligned}\right\}. \tag{i}$$

Let
$$z = \sum_1^m c_r y_r$$
be a linear function of the y's, which is changed into a multiple, λz, of itself by the substitution S. Then
$$\lambda \sum_1^m c_r y_r = \sum_1^{m-1} c_r y_{r+1} - c_m (a_m y_1 + a_{m-1} y_2 + \ldots + a_1 y_m);$$

so that the c's and λ are determined by

$$\begin{aligned}
0 &= c_m(\lambda+a_1)-c_{m-1},\\
0 &= c_m a_2 \qquad +c_{m-1}\lambda-c_{m-2},\\
0 &= c_m a_3 \qquad\qquad +c_{m-2}\lambda-c_{m-3},\\
&\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\\
0 &= c_m a_{m-1} \qquad\qquad\qquad +c_2\lambda-c_1,\\
0 &= c_m a_m \qquad\qquad\qquad\qquad +c_1\lambda.
\end{aligned}$$

The equation for λ, obtained by eliminating the c's, is

$$\lambda^m+a_1\lambda^{m-1}+a_2\lambda^{m-2}+\dots+a_{m-1}\lambda+a_m=0; \tag{ii}$$

and the ratios of the c's are given by

$$\frac{c_r}{c_m}=\lambda^{m-r}+a_1\lambda^{m-r-1}+\dots+a_{m-r-1}\lambda+a_{m-r},$$
$$(r=1,\,2,\,\dots,\,m-1).$$

These ratios are definite functions of the roots of (ii.), so that corresponding to any root λ of the equation there is a single linear function of the y's which is changed into λ times itself by S. If unity be written for c_m, this function is given by

$$z=\sum_1^m(\lambda^r+a_1\lambda^{r-1}+\dots+a_{r-1}\lambda+a_r)\,y_{m-r},$$

$$=\sum_1^m\lambda^{m-r}(y_r+a_1y_{r-1}+\dots+a_{r-1}y_1).$$

Consider next the linear function defined by

$$z^{(s)}=\frac{1}{s!}\frac{\partial^s z}{\partial\lambda^s}.\quad\dots\quad\dots$$

Its value is given by

$$z^{(s)}=\sum_{r=1}^{r=m-s}\binom{m-r}{s}\lambda^{m-r-s}(y_r+a_1y_{r-1}+\dots+a_{r-1}y_1);$$

and

$$\begin{aligned}
\lambda z^{(s)}+z^{(s-1)} &= \sum_{r=1}^{r=m-s}\binom{m-r}{s}\lambda^{m-r-s+1}(y_r+a_1y_{r-1}+\dots+a_{r-1}y_1)\\
&\quad+\sum_{r=1}^{r=m-s+1}\binom{m-r}{s-1}\lambda^{m-r-s+1}(y_r+a_1y_{r-1}+\dots+a_{r-1}y_1)\\
&= \sum_{r=1}^{r=m-s+1}\binom{m-r+1}{s}\lambda^{m-r-s+1}(y_r+a_1y_{r-1}+\dots+a_{r-1}y_1).
\end{aligned}$$

But, denoting as before the operation of the substitution S by an accent,

$$\begin{aligned} z'^{(s)} &= \sum_{r=1}^{r=m-s} \binom{m-r}{s} \lambda^{m-r-s} \left(y_r' + a_1 y_{r-1}' + \ldots + a_{r-1} y_1'\right) \\ &= \sum_{r=1}^{r=m-s} \binom{m-r}{s} \lambda^{m-r-s} \left(y_{r+} + a_1 y_r + \ldots + a_{r-1} y_2\right) \\ &= \sum_{r=2}^{r=m-s+1} \binom{m-r+1}{s} \lambda^{m-r-s+1} \left(y_r + a_1 y_{r-1} + \ldots + a_{r-2} y_2\right) \\ &= \lambda z^{(s)} + z^{(s-1)} - y_1 \sum_{r=1}^{r=m-s+1} \binom{m-r+1}{s} \lambda^{m-r-s+1} a_{r-1}. \end{aligned}$$

Hence, if λ is a $(s+1)$-ple root of the equation (ii), so that

$$\sum_{r=1}^{r=m-t+1} \binom{m-r+1}{t} \lambda^{m-r-t+1} a_{r-1} = 0, \quad (t = 0, 1, \ldots, s),$$

then

$$\begin{aligned} z' &= \lambda z, \\ z'^{(1)} &= \lambda z^{(1)} + z, \\ z'^{(2)} &= \lambda z^{(2)} + z^{(1)}, \\ &\ldots \quad \ldots \quad \ldots \\ z'^{(s)} &= \lambda z^{(s)} + z^{(s-1)}. \end{aligned}$$

So, if μ is a $(t+1)$-ple root of (ii), distinct from λ, and if

$$w = \sum_1^m \mu^{m-r} \left(y_r + a_1 y_{r-1} + \ldots + a_{r-1} y_1\right),$$

then

$$\begin{aligned} w' &= \mu w, \\ w'^{(1)} &= \mu w^{(1)} + w, \\ &\ldots \quad \ldots \quad \ldots \\ w'^{(t)} &= \mu w^{(t)} + w^{(t-1)}. \end{aligned}$$

That the m linear functions, z's, w's, ..., thus introduced are linearly independent may be shown as follows. Suppose, if possible, that a linear relation

$$\alpha z^{(s)} + \beta z^{(s-1)} + \ldots + \gamma w^{(t)} + \delta w^{(t-1)} + \ldots + \ldots = 0$$

connects them. Then

$$\alpha z'^{(s)} + \beta z'^{(s-1)} + \ldots + \gamma w'^{(t)} + \delta w'^{(t-1)} + \ldots = 0,$$

or $\alpha \left(\lambda z^{(s)} + z^{(s-1)}\right) + \beta \left(\lambda z^{(s-1)} + z^{(s-2)}\right) + \ldots + \gamma \left(\mu w^{(t)} + w^{(t-1)}\right)$

$$\ldots \quad \ldots \quad \ldots \quad \ldots + \delta . \left(\mu w^{(t-1)} + w^{(t-2)}\right) + \ldots = 0.$$

Hence $\quad \alpha z^{(s-1)} + \beta z^{(s-2)} + \ldots + \gamma (\lambda - \mu) w^{(t)} + \ldots = 0.$

Since $\lambda-\mu$ is different from zero, all the z's may thus be got rid of without the w's, &c., at the same time disappearing. Hence a linear relation such as that assumed above involves a linear relation among the functions of a single set, say the w's; and this would involve w itself being identically zero; which is not the case, since the coefficient of y_m in w is unity.

Finally then, the substitution S, so far as it affects the y's, *i.e.* the substitution (i), can be expressed in the form

$$\left.\begin{array}{lll} z' = \lambda z, & w' = \mu w, & u' = \nu u, \;\ldots \\ z'^{(1)} = \lambda z^{(1)} + z, & w'^{(1)} = \mu w^{(1)} + w, & \ldots\;\ldots\;\ldots \\ z'^{(2)} = \lambda z^{(2)} + z^{(1)}, & \ldots\;\ldots\;\ldots\;\ldots & \\ \ldots\;\ldots\;\ldots\;\ldots & w'^{(t)} = \mu w^{(t)} + w^{(t-1)}, & \\ z'^{(s)} = \lambda z^{(s)} + z^{(s-1)}, & & \end{array}\right\}, \tag{i)$'$}$$

where λ, μ, ν, ... are the distinct roots of

$$\lambda^m + a_1\lambda^{m-1} + a_2\lambda^{m-2} + \ldots + a_m = 0; \tag{ii)$'$}$$

$s+1$, $t+1$, ... are their multiplicities; and

$$z^{(p)} = \frac{1}{p!}\,\frac{\partial_n}{\partial\lambda^p}\sum_1^m \lambda^{m-r}\,(y_r + a_1 y_{r-1} + \ldots + a_{r-1}y_1),$$

$$w^{(q)} = \frac{1}{q!}\,\frac{\partial^q}{\partial\mu^q}\sum_1^m \mu^{m-r}\,(y_r + a_1 y_{r-1} + \ldots + a_{r-1}y_1),$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

The equations (i)$'$ are formally true whatever special values may be given to the A's. If the special values are such that z does not vanish identically, it is clear that no one of the functions $z^{(1)}, z^{(2)}, \ldots, z^{(s)}$ can vanish; and the same is true for the sets of functions in the other columns. It may, however, happen that, for some sets of values of the A's, $z, z^{(1)}, \ldots, z^{(k)}$ vanish identically, but that $z^{(k+1)}$ does not. No one of the functions $z^{(k+2)}, \ldots, z^{(s)}$ can then vanish; and for such special values the equations of the first column reduce to

$$\begin{aligned} z'^{(k+1)} &= \lambda z^{(k+1)}, \\ z'^{(k+2)} &= \lambda z^{(k+2)} + z^{(k+1)}, \\ \ldots\;\ldots &\;\ldots\;\ldots\;\ldots \\ z'^{(s)} &= \lambda z^{(s)} + z^{(s-1)}. \end{aligned}$$

It is clear that in no case can the replacing of the arbitrary A's by special values lead to a linear relation between non-vanishing functions occurring in different columns.

2. Suppose, now, that the z of the preceding paragraph has been actually calculated as a function of the original variables, or x's, and the original arbitrary coefficients, or A's. We may then write

$$z = B_1 z_1 + B_2 z_2 + \dots + B_p z_p,$$

where the z's are independent linear functions of the x's, and the B's are independent linear functions of the A's; with coefficients, in each case, which are functions of the coefficients a_{st} of the substitution S. In fact, if either the z's or the B's were not linearly independent, we could represent z as the sum of a smaller number of similar terms; and this process could be continued till the conditions in question are satisfied.

A similar form may be obtained for $z^{(1)}$, so that we may write

$$z^{(1)} = C_1 \zeta_1 + C_2 \zeta_2 + \dots + C_q \zeta_q.$$

Let the A's be replaced by n linear functions of themselves, of which $B_1, B_2, \dots, B_p$ are chosen for the first p. Then

$$C_r = \gamma_{r1} B_1 + \gamma_{r2} B_2 + \dots + \gamma_{rp} B_p + D_r,$$

where D_r is independent of $B_1, B_2, \dots, B_p$. Hence, if

$$z_a^{(1)} = \gamma_{ra} \zeta_1 + \gamma_{ra} \zeta_2 + \dots + \gamma_{qa} \zeta_q,$$

$$z^{(1)} = B_1 z_1^{(1)} + B_2 z_2^{(1)} + \dots + B_p z_p^{(1)} + D_1 \zeta_1 + D_2 \zeta_2 + \dots + D_q \zeta_q.$$

The function $D_1 \zeta_1 + \dots$ may again be expressed in the form

$$B_{p+1} z_{p+1}^{(1)} + B_{p+2} z_{p+2}^{(1)} + \dots + B_{p+p_1} z_{p+p_1}^{(1)},$$

where $B_{p+1}, B_{p+2}, \dots, B_{p+p_1}$ are linearly independent of each other, and of the preceding B's; and $z_{p+1}^{(1)}, z_{p+2}^{(1)}, \dots, z_{p+p_1}^{(1)}$ are linearly independent of each other.

If the A's are chosen so that

$$B_1 = B_2 = \dots = B_p = 0,$$

then the relation $$z'^{(1)} = \lambda z^{(1)} + z$$

gives $$B_{p+1} z'^{(1)}_{p+1} + \dots + B_{p+p_1} z'^{(1)}_{p+p_1} = \lambda (B_{p+1} z_{p+1}^{(1)} + \dots + B_{p+p_1} z_{p+p_1}^{(1)}).$$

It follows from this that $z_{p+1}^{(1)}, z_{p+2}^{(1)}, \ldots, z_{p+p_1}^{(1)}$ are linearly independent of $z_1^{(1)}, z_2^{(1)}, \ldots, z_p^{(1)}$. For an identical relation such as

$$Pz_1^{(1)}+Qz_2^{(1)}+\ldots+Rz_{p+1}^{(1)}+Sz_{p+2}^{(1)}+\ldots=0$$

involves $P(\lambda z_1^{(1)}+z_1)+Q(\lambda z_2^{(1)}+z_2)+\ldots+R\lambda z_{p+1}^{(1)}+S\lambda z_{p+2}^{(1)}+\ldots=0$,

or $$Pz_1+Qz_2+\ldots=0,$$

which can only hold if P, Q, ... are zero. This, moreover, shows that $z_1^{(1)}, z_2^{(1)}, \ldots, z_p^{(1)}$ are linearly independent of each other. It must, however, be noticed that $z_{p+1}^{(1)}, z_{p+2}^{(1)}, \ldots, z_{p+p_1}^{(1)}$ are not in general linearly independent of $z_1, z_2, \ldots, z_p$.

Continuing this process, we may now express each $z^{(r)}$, $(r=1, 2, \ldots, s)$, in the form

$$z^{(r)}=B_1z_1^{(r)}+\ldots+B_{p+1}z_{p+1}^{(r)}+\ldots+B_{p+p_1+1}z_{p+p_1+1}^{(r)}+\ldots$$
$$\ldots+B_{p+p_1+\ldots+p_r}z_{p+p_1+\ldots+p_r}^{(r)};$$

where, from the mode of expression, the B's are necessarily independent, while it may be shown as above that the $p+p_1+\ldots+p_r$ functions involved in $z^{(r)}$ are linearly independent.

In the general equations that correspond to the root λ of the equation (ii), let all the B's be made zero except $B_1, B_2, \ldots, B_p$. The system of $s+1$ equations, with p arbitrary coefficients,

$$\left.\begin{aligned} B_1z_1'+B_2z_2'+\ldots+B_pz_p' &= \lambda(B_1z_1+B_2z_2+\ldots+B_pz_p), \\ B_1z_1'^{(r)}+B_2z_2'^{(r)}+\ldots+B_pz_p'^{(r)} & \\ = \lambda(B_1z_1^{(r)}+\ldots+B_pz_p^{(r)}) &+ B_1z_1^{(r-1)}+\ldots+B_pz_p^{(r-1)}, \end{aligned}\right\}$$
$$(r=1, 2, \ldots, s), \quad \text{(iii)}$$

is thus obtained.

That the $p(s+1)$ functions contained in these equations, and therefore also the equations themselves, are linearly independent may be verified as before.

Next, make all the B's zero except $B_{p+1}, B_{p+2}, \ldots, B_{p+p_1}$. The system of s equations with p_1 arbitrary coefficients

$$\left.\begin{aligned} B_{p+1}z_{p+1}'^{(1)}+\ldots+B_{p+p_1}z_{p+p_1}'^{(1)} &= \lambda(B_{p+1}z_{p+1}^{(1)}+\ldots+B_{p+p_1}z_{p+p_1}^{(1)}), \\ B_{p+1}z_{p+1}'^{(r)}+\ldots+B_{p+p_1}z_{p+p_1}'^{(r)} & \\ = \lambda(B_{p+1}z_{p+1}^{(r)}+\ldots+B_{p+p_1}z_{p+p_1}^{(r)}) &+ B_{p+1}z_{p+1}^{(r-1)}+\ldots+B_{p+p_1}z_{p+p_1}^{(r-1)}, \end{aligned}\right\}$$
$$(r=2, 3, \ldots, s), \quad \text{(iv)}$$

is then obtained. These equations and the functions contained in them may, as before, be shown to be independent among themselves. They are, however, not necessarily independent of the functions contained in equations (iii).

The totality of the functions contained in equations (iv) arise from $\zeta^{(s)}$, or

$$B_{p+1} z_{p+1}^{(s)} + B_{p+2} z_{p+2}^{(s)} + \ldots + B_{p+p_1} z_{p+p_1}^{(s)},$$

and the results of operating repeatedly on this function with S. Hence, if

$$\zeta^{(s)} = \zeta_1^{(s)} + C_1 \zeta_{p+1}^{(s)} + C_2 \zeta_{p+2}^{(s)} + \ldots + C_{\varpi_1} \zeta_{p+\varpi_1}^{(s)},$$

where $C_1, C_2, \ldots, C_{\varpi_1}$ are independent linear functions of the B's; $\zeta_{p+1}^{(s)}, \zeta_{p+2}^{(s)}, \ldots, \zeta_{p+\varpi_1}^{(s)}$ independent linear functions of the variables; and $\zeta^{(s)} - \zeta_1^{(s)}$ denotes what $\zeta^{(s)}$ reduces to when all the functions in equations (iii) are made zero, the terms arising from $\zeta_1^{(s)}$ in equations (iv) destroy each other in virtue of equations (iii). Hence equations (iv) may be replaced by the system of s equations

$$\left.\begin{aligned} C_1 \zeta_{p+1}^{\prime(1)} + \ldots + C_{\varpi_1} \zeta_{p+\varpi_1}^{\prime(1)} &= \lambda \, (C_1 \zeta_{p+1}^{(1)} + \ldots + C_{\varpi_1} \zeta_{p+\varpi_1}^{(1)}), \\ C_1 \zeta_{p+1}^{\prime(r)} + \ldots + C_{\varpi_1} \zeta_{p+\varpi_1}^{\prime(r)} & \\ = \lambda \, (C_1 \zeta_{p+1}^{(r)} + \ldots + C_{\varpi_1} \zeta_{p+\varpi_1}^{(r)}) &+ C_1 \zeta_{p+1}^{(r-1)} + \ldots + C_{\varpi_1} \zeta_{p+\varpi_1}^{(r-1)}, \end{aligned}\right\}$$

$$(r = 2, 3, \ldots, s), \quad \text{(iv)}'$$

containing ϖ_1 arbitrary coefficients; in which the $\varpi_1 s$ variables are independent of each other, and of those in equations (iii).

Similarly, the system of $s-1$ equations with p_2 arbitrary coefficients resulting from making all the B's zero, except B_{p+p_1+1}, $B_{p+p_1+2}, \ldots, B_{p+p_1+p_2}$, may be replaced by a system of $s-1$ equations with ϖ_2 arbitrary coefficients in which the variables are linearly independent of each other and of those in equations (iii) and (iv)$'$.

This process may be continued till finally the general system of $s+1$ equations which correspond to the root λ of the equation (ii), viz.,

$$z' = \lambda z, \quad z'^{(1)} = \lambda z^{(1)} + z, \quad \ldots, \quad z'^{(s)} = \lambda z^{(s)} + z^{(s-1)},$$

is replaced by

a system	of	$s+1$	equations	of similar	form with	p	arbitraries,
,,	,,	s	,,	,,	,,	ϖ_1	,,
,,	,,	$s-1$	,,	,,	,,	ϖ_2	,,
...	...	...	...	...	...	...	...
,,	,,	1	,,	,,	,,	ϖ_s	,,

such that the

$$p(s+1)+\varpi_1 s+\varpi_2(s-1)+\ldots+\varpi_s = n_\lambda$$

variables entering in these equations are linearly independent. Moreover, whatever values the original arbitrary coefficients $A_1, A_2, \ldots, A_n$ may have, any one of the variables $z^{(r)}$ $(r = 0, 1, \ldots, s)$ entering in the general system can be expressed linearly in terms of these n_λ linearly independent variables.

From the general form obtained above for z it is clear that the integer p cannot be less than unity; but, in particular cases, any one or more of the integers $\varpi_1, \varpi_2, \ldots, \varpi_s$ may be zero.

The general system of $t+1$ equations corresponding to the root μ of equation (ii) may be similarly treated; and they will give rise to a similar set of systems of equations containing n_μ linearly independent variables. That these are linearly independent of the n_λ variables that arise from the root λ has already been shown (§ 1). Let

$$n' = n_\lambda + n_\mu + \ldots,$$

where the sum is extended to all the roots of equation (ii). Since the n' variables thus obtained are linearly independent, n' cannot be greater than n. On the other hand, since y_1, an absolutely arbitrary function of the original variables, can be expressed in terms of the z's, w's, ..., while the z's can be expressed in terms of the n_λ variables arising from the root λ, the w's in terms of the n_μ variables arising from the root μ, and so on, it follows that n' cannot be less than n. Hence

$$n' = n,$$

and the set of systems of equations, giving the effect of S upon the

$$n_\lambda + n_\mu + \ldots$$

new variables which have been constructed, is linearly equivalent

to the original equations

$$x'_s = \Sigma a_{st} x_t, \quad (s,\ t = 1,\ 2,\ \ldots,\ n).$$

The process by which the sets of quantities

$$\left.\begin{array}{llllll} \lambda, & s+1, & p, & \varpi_1, & \varpi_2, & \ldots, \ \varpi_s; \\ \mu, & t+1, & & \ldots & \ldots & \ldots; \\ \ldots & \ldots & \ldots & \ldots & & \ldots \end{array}\right\} \qquad \text{(v)}$$

have been derived from the equations of the substitution S is a definite and unique one. Any two substitutions A and B for which these sets of quantities are the same are conjugate in the general linear group. For, if $u, v, w, \ldots$ are the new variables which arise from A when reduced as above, and if $U, V, W, \ldots$ are the corresponding new variables which arise from B, then the substitution C, or

$$U = u, \quad V = v, \quad W = w, \quad \ldots,$$

is evidently such that

$$C^{-1}BC = A.$$

On the other hand, if C is any linear substitution, the sets of quantities in question are obviously the same for B and $C^{-1}BC$. Hence the necessary and sufficient conditions that two linear substitutions should be conjugate in the general linear group are that the sets of quantities (v) should be the same for both.

3. Since, as stated above, the method of reduction here explained is put forward as a practicable one, whenever the characteristic equation of the substitution can be actually solved, it seems proper to give a numerical example in which the reduction is really carried out.

Let S be

$$\begin{aligned} x'_1 &= -2x_1 - x_2 - x_3 + 3x_4 + 2x_5, \\ x'_2 &= -4x_1 + x_2 - x_3 + 3x_4 + 2x_5, \\ x'_3 &= x_1 + x_2 - 3x_4 - 2x_5, \\ x'_4 &= -4x_1 - 2x_2 - x_3 + 5x_4 + x_5, \\ x'_5 &= 4x_1 + x_2 + x_3 - 3x_4. \end{aligned}$$

The linear functions $y_1, y_2, y_3, \ldots$, as obtained by direct calculation, are given by the following table:—

	x_1	x_2	x_3	x_4	x_5
y_1	A_1	A_2	A_3	A_4	A_5
y_2	$-2A_1-4A_2+A_3$ $-4A_4+4A_5$	$-A_1+A_2+A_3$ $-2A_4+A_5$	$-A_1-A_2$ $-A_4+A_5$	$3A_1+3A_2-3A_3$ $+5A_4-3A_5$	$2A_1+2A_2-2A_3$ $+A_4$
y_3	$3A_1-A_2-2A_3$ $-A_4+A_5$	$-4A_1\quad+4A_3$ $-8A_4+4A_5$	$2A_1+2A_2-A_3$ $+2A_4-2A_5$	$3A_1+3A_2-3A_3$ $+7A_4-3A_5$	$-A_1-A_2+A_3$ $-5A_4+5A_5$
y_4	$-4A_1-12A_2+3A_3$ $-12A_4+12A_5$	$-12A_1-4A_2+12A_3$ $-24A_4+12A_5$	$-3A_1-3A_2+2A_3$ $-3A_4+3A_5$	$9A_1+9A_2-9A_3$ $+17A_4-9A_5$	$-3A_1-3A_2+3A_3$ $-15A_4+11A_5$
y_5	$5A_1-11A_2-4A_3$ $-11A_4+11A_5$	$-32A_1-16A_2+32A_3$ $-64A_4+32A_5$	$4A_1+4A_2-3A_3$ $+4A_4-4A_5$	$15A_1+15A_2-15A_3$ $+31A_4-15A_5$	$-17A_1-17A_2+17A_3$ $-49A_4+33A_5$

No linear relation connects the first two, three, or four y's, but

between the first five the relation

$$y_5-2y_4-3y_3+4y_2+4y_1=0$$

holds. The equation for λ is therefore

$$\lambda^4-2\lambda^3-3\lambda^2+4\lambda+4\equiv(\lambda+1)^2(\lambda-2)^2=0.$$

The coefficients in z, w, $z^{(1)}$, $w^{(1)}$ are given by the table:—

	$\lambda=-1$	$\lambda=2$
$\lambda^3-2\lambda^2-3\lambda+4$	4	-2
$\lambda^2-2\lambda-3$	0	-3
$\lambda-2$	-3	0
$3\lambda^2-4\lambda-3$	4	1
$2\lambda-2$	-4	2

Hence $z=4y_1-3y_3+y_4$

$$=-9(A_1+A_2-A_3+A_4-A_5)(x_1+x_3)=-9B_1(x_1+x_3),$$

$$z^{(1)}=4y_1-4y_2+y_3$$

$$=B_1(15x_1+6x_3-9x_4-9x_5)+9A_3(x_1+x_3),$$

$$w=-2y_1-3y_2+y_4$$

$$=-9(A_1+A_2-A_3+2A_4-A_5)(x_2+x_5)=-9C_1(x_2+x_5),$$

$$w^{(1)}=y_1+2y_2+y_3$$

$$=-9(A_2+A_4-A_5)x_1-(6C_1-9A_2)x_2$$
$$+9C_1x_4+(3C_1-9A_4+9A_5)x_5$$

$$=C_1(-6x_2+9x_4+3x_5)-9(A_2+A_4-A_5)(x_1-x_2)$$
$$-9(A_4-A_5)(x_2+x_5).$$

Finally then, if
$$\xi_1 = -9\,(x_1+x_3),$$
$$\xi_2 = 15x_1+6x_3-9x_4-9x_5,$$
$$\xi_3 = -9\,(x_2+x_5),$$
$$\xi_4 = -6x_2+9x_4+3x_5,$$
$$\xi_5 = x_1-x_2,$$

the substitution S can be expressed in the form
$$\xi_1' = -\xi_1,$$
$$\xi_2' = \xi_1-\xi_2,$$
$$\xi_3' = 2\xi_3,$$
$$\xi_4' = \xi_3+2\xi_4,$$
$$\xi_5' = 2\xi_5.$$

The linear functions which enter in a canonical form of a linear substitution are never unique, unless the roots of the characteristic equation are all distinct. It is not then to be expected that a definite process for reducing a substitution to its canonical form will always give the reducing equations in as simple a form as possible. In fact, in the above example, the simpler relations

$$\xi_1 = x_1+x_3,\quad \xi_2 = x_3+x_4+x_5,\quad \xi_3 = x_2+x_5,\quad \xi_4 = -x_4-x_5,\quad \xi_5 = x_1-x_2$$

will equally well reduce the substitution.

4. If the substitution which it is proposed to reduce happens to be a substitution of finite period, the application of the method is particularly simple. The substitution (i)′ of § 1 can, in fact, be of finite period only when $s = t = \ldots = 0$. The roots of the equation (ii)′ are therefore in this case all distinct, and the calculation of

$$\sum_1^m (\lambda^r+a_1\lambda^{r-1}+\ldots+a_r)\, y_{m-r}$$

for each root of the equation at once gives the reducing substitution.

Note to the foregoing paper. *By* H. F. BAKER, F.R.S.

Received January 30th, 1899.

1. Let $$M = \begin{pmatrix} c_{11} & c_{12} & \dots \\ c_{21} & c_{22} & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

be a matrix, such that the determinant Δ, of the matrix D, given by

$$D = M - \theta = \begin{pmatrix} c_{11}-\theta & c_{12} & \dots \\ c_{21} & c_{22}-\theta & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

contains a factor $(\theta-\theta_1)^l$. Suppose that the highest power of this factor which divides the determinants of all first minors of D is $(\theta-\theta_1)^{l_1}$.

It is required to show that the equation satisfied by the matrix M can be obtained by forming the product of all the factors

$$(\theta-\theta_1)^{l-l_1}$$

for all the different roots of $\Delta = 0$, and then replacing θ by M.

It follows, since l is always greater than l_1, that when all the roots of the equation satisfied by M are known, the equation $\Delta = 0$ can be solved.

Let $$a_{ii} = c_{ii}-\theta, \quad a_{rs} = c_{rs}$$

be the elements of the matrix D, and A_{rs} the determinants of its first minors. Then, as a product of matrices,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \begin{vmatrix} A_{11}/\Delta & A_{21}/\Delta & \dots \\ A_{12}/\Delta & \dots & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \dots & \dots & \dots \end{vmatrix} = 1;$$

and therefore $$D^{-1} = \left(\frac{A_{sr}}{\Delta}\right),$$

where the right side denotes the matrix which is the second factor in the product just written.

Since there are only n^2 elements in the matrices of n rows and columns, it follows that not more than n^2 of the successive powers of the matrix M can be linearly independent; suppose that we have

$$\psi(M) = M^m + C_1 M^{m-1} + \dots + C_m = 0,$$

M not satisfying an equation of lower order, $C_1, \ldots, C_m$ being numerical quantities which are functions of the elements in M; let

$$\chi(M, \theta) = M^{m-1} + (\theta + C_1) M^{m-2} + (\theta^2 + C_1\theta + C_2) M^{m-3} + \ldots$$
$$\ldots + (\theta^{m-1} + C_1\theta^{m-2} + \ldots + C_{m-1});$$

then the matrix $\chi(M, \theta)$ cannot be zero for any value of θ, since else M would satisfy an equation of lower than the m^{th} order.

We immediately verify the identity

$$(M-\theta)\,\chi(M, \theta) = \psi(M) - \psi(\theta) = -\psi(\theta),$$

so that
$$\psi(\theta) D^{-1} = -\chi(M, \theta)$$

is a matrix of which every element is an integral function of θ.

If, then, $\theta - \theta_1$ is an l times repeated factor of Δ, and the greatest power of $\theta - \theta_1$ which divides the first minors A_{sr} be $(\theta-\theta_1)^{l_1}$, it follows, since each of the quantities $\psi(\theta)\dfrac{A_{sr}}{\Delta}$ is an integral function of θ, that $\psi(\theta)$ divides by $(\theta-\theta)^{l-l_1}$.

Conversely, if $\theta - \phi$ be an a times repeated factor of $\psi(\theta)$, then the determinant of the matrix $M - \phi$ must vanish; for otherwise we should have

$$\chi(M, \phi) = -(M-\phi)^{-1}\psi(\phi) = 0;$$

suppose $\theta - \phi$ is a λ times repeated factor of Δ, and a λ_1 times repeated factor of the greatest common divisor (in regard to θ) of the first minors A_{sr}; then, as before, $(\theta-\phi)^{\lambda-\lambda_1}$ divides $\psi(\theta)$, or $a \overset{=}{>} \lambda - \lambda_1$. If $a > \lambda - \lambda_1$, the elements of the matrix

$$(\theta-\phi)^{-(a-\lambda+\lambda_1)}\psi(\theta) D^{-1} = -(\theta-\phi)^{-(a-\lambda+\lambda_1)}\chi(M, \theta)$$

would be integral functions of θ, and therefore

$$\chi(M, \varphi) = 0,$$

which is impossible.

Hence
$$a = \lambda - \lambda_1.$$

This proof is modified from Frobenius, *Crelle*, LXXXIV. (1878), pp. 12, 26.

2. With this notation the fundamental solutions $z, z^{(1)}, \ldots, z^{(s)}$ of the foregoing paper (p. 184) are almost intuitive. Since we subtract two matrices by subtracting all elements of one from the corresponding elements of the other, we can differentiate a matrix by differentiating all its elements. Hence, if K, L be matrices,

$$d(KL) = dK \, . \, L + K \, . \, dL.$$

Thence, from $\psi(\phi) = -D \cdot \chi(M, \theta)$,

$$\psi'(\theta) = \chi(M, \theta) - D\frac{\partial}{\partial\theta}\chi(M, \theta),$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$\psi^{(s)}(\theta) = s\frac{\partial^{s-1}}{\partial\theta^{s-1}}\chi(M, \theta) - D\frac{\partial^s}{\partial\theta^s}\chi(M, \theta).$$

Suppose $\theta = \theta_1$ is an $(s+1)$ times $= (l-l_1)$ times repeated root of $\psi(\theta)$, and, with $A_1, \dots, A_n$ arbitrary, put

$$z^{(s)} = \frac{1}{s!}\left(\frac{\partial^s}{\partial\theta_1^s}\chi(M, \theta_1)\between A_1, \dots, A_n \between x_1, \dots, x_n\right), \tag{I.}$$

which is a bilinear form, the quantity

$$\frac{\partial^s}{\partial\theta_1^s}\chi(M, \theta_1)$$

being a matrix; hence we immediately have from these equations the following,

$$D_1 z = 0, \quad D_1 z^{(1)} = z, \quad \dots, \quad D_1 z^{(s)} = z^{(s-1)},$$

where $D_1 = M - \theta_1$; thus (*cf.* p. 185)

$$Mz = \lambda z, \quad Mz^{(1)} = \lambda z^{(1)} + z, \quad \dots, \quad Mz^{(s)} = \lambda z^{(s)} + z^{(s-1)}.$$

In the foregoing paper we have, if Ω be the matrix of the substitution considered,

$$\begin{aligned} y_1 &= A_1 x_1 + \dots + A_n x_n, \\ y_2 &= A_1 x_1' + \dots + A_n x_n' \\ &= (A_1, \dots, A_n)\,\Omega(x_1, \dots, x_n) \\ &= \Omega(x_1, \dots, x_n)(A_1, \dots, A_n) \\ &= \overline{\Omega}(A_1, \dots, A_n \between x_1, \dots, x_n), \end{aligned}$$

and in general $\quad y_{m-r} = \overline{\Omega}^{(m-r-1)}(A_1, \dots, A_n \between x_1, \dots, x_n)$,

where $\overline{\Omega}$ is obtained from Ω by interchanging rows and columns. The function z in the paper is

$$\begin{aligned} z &= \sum_1^m (\theta_1^r + C_1\theta_1^{r-1} + \dots + C_{r-1}\theta_1 + C_r)\, y_{m-r} \\ &= \sum_1^m \left((\theta_1^r + C_1\theta_1^{r-1} + \dots + C_{r-1}\theta_1 + C_r)\,\overline{\Omega}^{(m-r-1)} \between A_1, \dots, A_n \between x_1, \dots, x_n\right), \end{aligned}$$

agreeing entirely with z as now defined, if

$$\overline{\Omega} = M.$$

Thus the functions $z, z^{(1)}, \ldots, z^{(s)}$ of the paper are those given by formula (I.) above, provided the matrix of the linear substitution of the paper be the transposed of that here denoted by M.

Thursday, February 9th, 1899.

Lt.-Col. A. J. C. CUNNINGHAM, R.E., Vice-President, in the Chair.

Eighteen members present.

Mr. Umes Chandra Ghosh, M.A., Lecturer in Mathematics, Muir Central College, Allahabad, was elected a member, and Mr. E. W. Barnes was admitted into the Society.

Mr. Berry read a paper entitled "Note on a Case of Divisibility of a Function of Two Variables by another Function."

The following papers were also read:—

The Scattering of Electric Waves by an Insulating Sphere: Mr. A. E. H. Love. Dr. Larmor and Prof. Lamb made a few remarks on the subject of the paper.

Groups of Order p^3q: Mr. A. E. Western.

The Irreducible Concomitants of any number of Binary Quartics: Mr. A. Young.

Mr. Western also communicated a paper by Dr. L. E. Dickson entitled "The Group of Linear Homogeneous Substitutions on mq Variables which is defined by a certain Invariant."

The remaining papers were read in abstract, viz.:—

A Note on Minimal Surfaces and On some Solutions of $\nabla^2 v = 0$: Mr. T. J. I'A. Bromwich.

On the Complete System of Multilinear Differential Covariants of a Single Pfaffian Expression, and of a Set of Pfaffian Expressions: Mr. J. Brill.

The Jacobian Locus in Hyper-geometry: Prof. Schoute.

The following presents were made to the Library :—

D. Aitoff.—"Problèmes de Géométrie élémentaire groupés d'après les méthodes à employer pour les résoudre," traduit du Russe (de Ivan Alexandroff), 8vo; Paris, 1899.

G. Oltramare.—"Leçons sur le Calcul de Généralisation," 8vo; Paris, 1899.

(These two books presented by the Publisher, M. A. Hermann; a second copy of the latter work has also been received from the Author.)

"Periodico di Matematica," Serie 2, Vol. I., Fasc. 1-4; and "Supplemento" (Anno 11, Fasc. 1, 1898). Presented by Dr. G. Lazzeri, Leghorn.

"Report of the Superintendent of the United States Naval Observatory, for the year ending June 30th, 1898"; Washington, 1898.

J. F. Igurbide.—"La Nouvelle Science Géométrique (Géométrie du Cercle)," 8vo; Barcelona, 1898. From the Author.

"L'Enseignement Mathématique," Revue Internationale, Paris. Two copies of No. 1 from the Editors, MM. Carré and Naud.

"Tokyo Imperial University Calendar," 1897-8.

"Educational Times," February, 1899. From the Publisher.

"Indian Engineering," Vol. XXIV., Nos. 26, 27, Dec. 24, 1898; Vol. XXV., Nos. 1, 2, January 14, 1899.

The following, bound in half calf, were presented by Mr. Tucker :—

Weber, H.—"Lehrbuch der Algebra," 2e Auflage, Bd. I., 8vo; Braunschweig, 1898.

Grassmann, H.—"Gesammelte mathematische und physikalische Werke," Bd. I., Theil 2: "Die Ausdehnungslehre von 1862 in Gemeinschaft mit H. Grassmann dem jüngerer, herausgegeben von Fr. Engel," 8vo; Leipzig, 1896.

Koll, O.—"Die Theorie der Beobachtungsfehler und die Methode der kleinsten Quadrate," 8vo; Berlin, 1893.

The following exchanges were received :—

"Proceedings of the Royal Society," Vol. LXIV., No. 406, 1899.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXII., St. 12, 1898; Bd. XXIII., St. 1; Leipzig, 1899.

"Bulletin de la Société Mathématique de France," Tome XXVI., No. 10 et dernier; Paris, 1898.

"Bulletin of the American Mathematical Society," Series 2, Vol. V., No. 4; January, 1899.

"Monatshefte für Mathematik und Physik," Jahrgang X., Vierteljahr 1; 1899.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. IV., Fasc. 12; Napoli, December, 1898.

"Journal für die reine und angewandte Mathematik," Bd. CXX., Heft 1; Berlin, 1899.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 2, Vol. VII., Fasc. 12, 1898; Sem. 1, Vol. VIII., Fasc. 1; Roma, 1899.

"Nyt Tidsskrift for Matematik," A, Aarg. 9, Nr. 5; B, Aarg. 9, Nrs. 3, 4; Copenhagen, 1898.

"Revue Semestrielle des Publications Mathématiques," Tome VII., Pt. 1, Av.-Oct., 1898; 1899.

"Journal of the Institute of Actuaries," Vol. xxxiv., Pt. 4, No. 192 ; 1898.
"Wiskundige Opgaven," Deel vii., St. 6 ; Amsterdam, 1899.
"Proceedings of the Physical Society," Vol. xvi., Pt. 4 ; January, 1899.
"Proceedings of the Cambridge Philosophical Society," Vol. x., Pt. 1 ; 1899.
"Transactions of the Cambridge Philosophical Society," Vol. xvii., Pt. 2 ; 1899.

The Group of Linear Homogeneous Substitutions on mq Variables which is defined by the Invariant

$$\phi \equiv \sum_{i=1}^{m} \xi_{i1}\xi_{i2}\dots\xi_{iq}.$$

By L. E. Dickson, Ph.D. Received January 30th, 1899. Communicated by A. E. Western, M.A., February 9th, 1899.

1. For the case $q=2$, the continuous group leaving ϕ invariant has been fully studied by Sophus Lie; also the discontinuous group of linear substitutions in a Galois field leaving ϕ invariant has been recently studied by the writer in the *Proc. Lond. Math. Soc.* We suppose $q>2$ in the present paper.

2. For the sake of clearness, the method of investigation is first illustrated in the simple case $m=1$, $q=4$. The conditions imposed upon the substitution

$$S: \ \xi_i' = \sum_{k=1}^{4} a_{ik}\xi_k \quad (i=1, \dots, 4),$$

in order that it leave $\xi_1\xi_2\xi_3\xi_4$ invariant, are as follows:—

(1) $$a_{1j}a_{2j}a_{3j}a_{4j} = 0 \quad (j=1, \dots, 4),$$

(2) $$a_{1j}a_{2j}a_{3j}a_{4k} + a_{1j}a_{2j}a_{4j}a_{3k} + a_{1j}a_{3j}a_{4j}a_{2k} + a_{2j}a_{3j}a_{4j}a_{1k} = 0$$
$$(j, k = 1, \dots, 4;\ j \neq k),$$

(3) $$a_{1j}a_{2j}a_{3k}a_{4k} + a_{1j}a_{3j}a_{2k}a_{4k} + a_{1j}a_{4j}a_{2k}a_{3k} + a_{2j}a_{3j}a_{1k}a_{4k}$$
$$+ a_{2j}a_{4j}a_{1k}a_{3k} + a_{3j}a_{4j}a_{1k}a_{2k} = 0,$$

(4) $$\Sigma\, a_{1j}a_{2j}a_{3k}a_{4l} = 0$$
$$(j, k, l = 1, \dots, 4;\ j \neq k \neq l),$$

the sum extending over 12 terms given by permuting 1, 2, 3, 4.

$$\Sigma a_{i1} a_{j2} a_{k3} a_{l4} = 1, \tag{5}$$

the sum extending over 24 terms obtained by giving the set (i, j, k, l) every permutation of (1, 2, 3, 4).

Multiplying equation (3) by 2, we obtain an equation of the form (4) for $k = l$; multiplying (2) by 3, we obtain an equation of the form (4) for $l = j$. Hence, if j and k be any two distinct integers chosen from 1, 2, 3, 4, we have the set of four equations given by $l = 1, 2, 3, 4$:

$$\begin{aligned}
&(a_{1j} a_{2j} a_{3k} + a_{1j} a_{3j} a_{2k} + a_{2j} a_{3j} a_{1k})\, a_{4l} \\
+ &(a_{1j} a_{2j} a_{4k} + a_{1j} a_{4j} a_{2k} + a_{2j} a_{4j} a_{1k})\, a_{3l} \\
+ &(a_{1j} a_{3j} a_{4k} + a_{1j} a_{4j} a_{3k} + a_{3j} a_{4j} a_{1k})\, a_{2l} \\
+ &(a_{2j} a_{3j} a_{4k} + a_{2j} a_{4j} a_{3k} + a_{3j} a_{4j} a_{2k})\, a_{1l} = 0.
\end{aligned}$$

Regarding the quantities in the parentheses as the unknowns, the determinant of the coefficients is seen to equal the determinant

$$| a_{ik} | \quad (i, k = 1, \ldots, 4)$$

of the substitution S, and is therefore assumed to be not zero. Hence the quantities in parentheses are all zero. But the above equations hold true if $j = k$. Indeed, for $l = j$, it is obtained by multiplying (1) by 12; for $l \neq j$, it is given by multiplying (2) by 3. We have therefore the result

$$a_{1j} a_{2j} a_{3k} + a_{1j} a_{3j} a_{2k} + a_{2j} a_{3j} a_{1k} = 0 \quad (j, k = 1, \ldots, 4),$$

with three similar equations obtained from it by replacing the index 1 by 4, or 2 by 4, or, finally, 3 by 4. It follows that the products $a_{1j} a_{2j}$, $a_{1j} a_{3j}$, $a_{2j} a_{3j}$ are all zero; indeed not every determinant of the matrix

$$\begin{vmatrix} a_{31} & a_{21} & a_{11} \\ a_{32} & a_{22} & a_{12} \\ a_{33} & a_{23} & a_{13} \\ a_{34} & a_{24} & a_{14} \end{vmatrix}$$

can be zero, since the determinant of S does not vanish.

Replacing the indices 1, 2, 3 in turn by 4, we find that also the products $a_{4j} a_{2j}$, $a_{4j} a_{3j}$, $a_{4j} a_{1j}$ vanish. Hence three of the quantities

$a_{1j}, a_{2j}, a_{3j}, a_{4j}$ vanish for every $j = 1, \ldots, 4$. In order that the determinant of S be not zero, the four non-vanishing coefficients must lie in distinct columns (as well as in distinct rows). We have therefore the result:

THEOREM.—*Every quaternary linear homogeneous substitution leaving $\xi_1\xi_2\xi_3\xi_4$ invariant can be generated by the substitutions $(\xi_i\xi_j)$ together with the following:—*

$$\xi_i' = a_{ii}\xi_i \quad (i = 1, \ldots, 4),$$

$$a_{11}a_{22}a_{33}a_{44} = 1.$$

The result holds for continuous groups, collineation groups, or for groups in any Galois field.

3. We will explain for the case $m = 1$, $q = 4$ the use of a symbol employed for brevity in the general case. The 24 terms of the left member of (5) are given by the expansion of the determinant

$$| a_{rs} | \quad (r, s = 1, \ldots, 4),$$

if all signs be taken positive. We therefore write (5) thus:

$$(5) \quad \left\{ \begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{matrix} \right\} = 1.$$

In the same notation relations (4), (3), (2), (1) become respectively

$$\tfrac{1}{2} \left\{ \begin{matrix} a_{1j} & a_{1j} & a_{1k} & a_{1l} \\ a_{2j} & a_{2j} & a_{2k} & a_{2l} \\ a_{3j} & a_{3j} & a_{3k} & a_{3l} \\ a_{4j} & a_{4j} & a_{4k} & a_{4l} \end{matrix} \right\} = 0, \qquad \tfrac{1}{4} \left\{ \begin{matrix} a_{1j} & a_{1j} & a_{1k} & a_{1k} \\ a_{2j} & a_{2j} & a_{2k} & a_{2k} \\ a_{3j} & a_{3j} & a_{3k} & a_{3k} \\ a_{4j} & a_{4j} & a_{4k} & a_{4k} \end{matrix} \right\} = 0,$$

$$\tfrac{1}{6} \left\{ \begin{matrix} a_{1j} & a_{1j} & a_{1j} & a_{1k} \\ a_{2j} & a_{2j} & a_{2j} & a_{2k} \\ a_{3j} & a_{3j} & a_{3j} & a_{3k} \\ a_{4j} & a_{4j} & a_{4j} & a_{4k} \end{matrix} \right\} = 0, \qquad \tfrac{1}{24} \left\{ \begin{matrix} a_{1j} & a_{1j} & a_{1j} & a_{1j} \\ a_{2j} & a_{2j} & a_{2j} & a_{2j} \\ a_{3j} & a_{3j} & a_{3j} & a_{3j} \\ a_{4j} & a_{4j} & a_{4j} & a_{4j} \end{matrix} \right\} = 0.$$

4. Consider a general substitution on mq indices,

$$S: \quad \xi'_{ij} = \sum_{\substack{k=1\dots m\\ l=1\dots q}} a_{kl}^{ij} \xi_{kl} \quad \begin{pmatrix} i = 1 \dots m \\ j = 1 \dots q \end{pmatrix}.$$

It transforms ϕ into the function

$$\phi' \equiv \sum_{i=1}^{m} \left\{ \left(\sum_{k_1, l_1} a_{k_1 l_1}^{i\,1} \xi_{k_1 l_1} \right) \left(\sum_{k_2, l_2} a_{k_2 l_2}^{i\,2} \xi_{k_2 l_2} \right) \dots \left(\sum_{k_q, l_q} a_{k_q l_q}^{i\,q} \xi_{k_q l_q} \right) \right\},$$

where $k_1, \dots, k_q = 1, \dots, m$; $l_1, \dots, l_q = 1, \dots, q$ independently.

Employing the symbol explained above, ϕ' takes the form

$$\sum \left[\frac{1}{C} \sum_{i=1}^{m} \begin{Bmatrix} a_{k_1 l_1}^{i\,1} & a_{k_2 l_2}^{i\,1} & \dots & a_{k_q l_q}^{i\,1} \\ a_{k_1 l_1}^{i\,2} & a_{k_2 l_2}^{i\,2} & \dots & a_{k_q l_q}^{i\,2} \\ \dots & \dots & \dots & \dots \\ a_{k_1 l_1}^{i\,q} & a_{k_2 l_2}^{i\,q} & \dots & a_{k_q l_q}^{i\,q} \end{Bmatrix} \xi_{k_1 l_1} \xi_{k_2 l_2} \dots \xi_{k_q l_q} \right]$$

summed for every *combination* of q pairs of integers (k_1, l_1), (k_2, l_2), ..., (k_q, l_q), the k's being selected from the integers $1, \dots, m$, and the l's from $1, \dots, q$, repetitions allowed. To determine the value of C for a given combination of q pairs (k_i, l_i), we separate them into τ sets

$$(6) \begin{cases} (k_1, l_1) = (k_2, l_2) = \dots = (k_{t_1}, l_{t_1}), \\ (k_{t_1+1}, l_{t_1+1}) = \dots = (k_{t_1+t_2}, l_{t_1+t_2}), \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ (k_{t_1+t_2+\dots+t_{\tau-1}+1}, l_{t_1+t_2+\dots+t_{\tau-1}+1}) = \dots = (k_{t_1+t_2+\dots+t_\tau}, l_{t_1+t_2+\dots+t_\tau}), \end{cases}$$

those in different sets being distinct pairs. We readily see that

$$C = \frac{(t_1 + t_2 + \dots + t_\tau)!}{t_1!\, t_2! \dots t_\tau!}, \quad t_1 + t_2 + \dots + t_\tau = q.$$

The identity $\phi' = \phi$ therefore requires the following relations :—*

$$(7) \sum_{i=1}^{m} \begin{Bmatrix} a_{j1}^{i1} & a_{j2}^{i1} & \dots & a_{jq}^{i1} \\ a_{j1}^{i2} & a_{j2}^{i2} & \dots & a_{jq}^{i2} \\ \dots & \dots & & \dots \\ a_{j1}^{iq} & a_{j2}^{iq} & \dots & a_{jq}^{iq} \end{Bmatrix} = 1 \quad (j = 1, \dots, m),$$

* The numerical factor in (8) must be retained when working with linear groups in certain Galois fields; indeed, the reciprocal of this factor might be zero, in which case it could not be omitted.

$$(8)\quad \frac{t_1!\,t_2!\,\ldots\,t_r!}{q!}\sum_{i=1}^{m}\begin{Bmatrix} a^{i\,1}_{k_1 l_1} & a^{i\,1}_{k_2 l_2} & \ldots & a^{i\,1}_{k_q l_q} \\ \ldots & \ldots & \ldots & \ldots \\ a^{i\,q}_{k_1 l_1} & a^{i\,q}_{k_2 l_2} & \ldots & a^{i\,q}_{k_q l_q} \end{Bmatrix} = 0,$$

holding for every combination of q pairs $(k_1, l_1), \ldots, (k_q, l_q)$, except the combinations $(k_1, 1), (k_1, 2), \ldots, (k_1, q)$, that can be formed from $(1, 1), \ldots, (1, q), \ldots, (m, 1), \ldots, (m, q)$. The numerical factor is determined by (6).

5. Consider the totality of relations (8) in which $(k_1, l_1) = (k_2, l_2)$. Multiply each by the factor $\frac{1}{2}\frac{q!}{t_1!\,\ldots\,t_r!}$, which will be an integer since the pair (k_1, l_1) is of multiplicity at least 2. Group together the mq relations in which $k_1, k_2, \ldots, k_{q-1}$; $l_1, l_2, \ldots, l_{q-1}$ have arbitrarily fixed values, while k_q runs from 1 to m, l_q from 1 to q. Note that we tacitly assume that $q > 2$. We may expand the general one of these mq relations into the form

$$\sum_{i=1}^{m}\left[\tfrac{1}{2}\begin{Bmatrix} a^{i\,1}_{k_1 l_1} & \ldots\ldots & a^{i\,1}_{k_{q-1} l_{q-1}} \\ \ldots & \ldots & \ldots \\ a^{i\,q-1}_{k_{q-1} l_{q-1}} & \ldots & a^{i\,q-1}_{k_{q-1} l_{q-1}} \end{Bmatrix} a^{i\,q}_{k_q l_q} + \ldots + \tfrac{1}{2}\begin{Bmatrix} a^{i\,2}_{k_1 l_1} & \ldots & a^{i\,2}_{k_{q-1} l_{q-1}} \\ \ldots & \ldots & \ldots \\ a^{i\,q}_{k_1 l_1} & \ldots & a^{i\,q}_{k_{q-1} l_{q-1}} \end{Bmatrix} a^{i\,1}_{k_q l_q}\right]$$
$$= 0.$$

In these mq equations, given by $k_q = 1, \ldots, m$; $l_q = 1, \ldots, q$, the mq quantities in brackets are the same throughout, and may be regarded as the unknown quantities. The determinant of their coefficients

$$\left(a^{i\,j}_{k_q l_q}\right)\quad \begin{pmatrix} i,\ k_q = 1, \ldots, m \\ j,\ l_q\ = 1, \ldots, q \end{pmatrix}$$

is not zero, being the determinant of the substitution S. Hence the mq unknowns are all zero, viz.,

$$(9)\quad \tfrac{1}{2}\begin{Bmatrix} a^{i\,a_1}_{k_1 l_1} & a^{i\,a_1}_{k_2 l_2} & \ldots & a^{i\,a_1}_{k_{q-1} l_{q-1}} \\ \ldots & \ldots & \ldots & \ldots \\ a^{i\,a_{q-1}}_{k_1 l_1} & a^{i\,a_{q-1}}_{k_2 l_2} & \ldots & a^{i\,a_{q-1}}_{k_{q-1} l_{q-1}} \end{Bmatrix} = 0,$$

where $a_1, a_2, \ldots, a_{q-1}$ are distinct integers chosen arbitrarily from $1, 2, \ldots, q$, and $i, k_1, k_3, \ldots, k_{q-1} = 1, \ldots, m$; $l_1, l_3, \ldots, l_{q-1} = 1, \ldots, q$ *independently*. *If* $q-1 = 2$, we have the result (11′) below.

If $q-1>2$, we consider the relations (9) in which $i, k_1, k_3, \ldots, k_{q-2}$; $l_1, l_3, \ldots, l_{q-2}$ have arbitrarily fixed values, while k_{q-1} runs from 1 to m, l_{q-1} from 1 to q. The general one of these mq relations may be expanded into the form

$$\tfrac{1}{2}\begin{Bmatrix} a_{k_1 l_1}^{i\ a_1} & \ldots & a_{k_{q-2} l_{q-2}}^{i\ a_1} \\ \ldots & \ldots & \ldots \quad \ldots \\ a_{k_1 l_1}^{i\ a_{q-2}} & \ldots & a_{k_{q-2} l_{q-2}}^{i\ a_{q-2}} \end{Bmatrix} a_{k_{q-1} l_{q-1}}^{i\ \ a_{q-1}} + \ldots + \tfrac{1}{2}\begin{Bmatrix} a_{k_1 l_1}^{i\ a_2} & \ldots & a_{k_{q-2} l_{q-2}}^{i\ a_2} \\ \ldots & \ldots & \ldots \quad \ldots \\ a_{k_1 l_1}^{i\ a_{q-1}} & \ldots & a_{k_{q-2} l_{q-2}}^{i\ a_{q-1}} \end{Bmatrix} a_{k_{q-1} l_{q-1}}^{i\ \ a_1} = 0.$$

Consider as unknowns the $q-1$ quantities in brackets. The matrix of the coefficients is composed of $q-1$ rows of the determinant of S. Hence not every determinant of order $q-1$ in the matrix is zero. Hence the $q-1$ unknowns are all zero, viz.,

$$(10) \qquad \tfrac{1}{2}\begin{Bmatrix} a_{k_1 l_1}^{i\ b_1} & a_{k_2 l_2}^{i\ b_1} & \ldots & a_{k_{q-2} l_{q-2}}^{i\ b_1} \\ \ldots & \ldots & \ldots & \ldots \quad \ldots \quad \ldots \\ a_{k_1 l_1}^{i\ b_{q-2}} & a_{k_2 l_2}^{i\ b_{q-2}} & \ldots & a_{k_{q-2} l_{q-2}}^{i\ b_{q-2}} \end{Bmatrix} = 0,$$

where $b_1, b_2, \ldots, b_{q-2}$ are distinct integers chosen arbitrarily from $1, 2, \ldots, q$, while $i, k_1, k_3, \ldots, k_{q-2} = 1, 2, \ldots, m$; $l_1, l_3, \ldots, l_{q-2} = 1, 2, \ldots, q$ independently. If $q-2=2$, we have the equations (11′) below.

If $q-2>2$, we proceed as before. Finally, we reach the result

$$(11') \qquad \tfrac{1}{2}\begin{Bmatrix} a_{k_1 l_1}^{i\ c} & a_{k_2 l_2}^{i\ c} \\ a_{k_1 l_1}^{i\ d} & a_{k_2 l_2}^{i\ d} \end{Bmatrix} = 0,$$

where c, d are distinct integers chosen arbitrarily from $1, 2, \ldots, q$. Since $(k_1, l_1) = (k_2, l_2)$, we have the result

$$(11) \qquad a_{kl}^{i\,c}\, a_{kl}^{i\,d} = 0 \quad \begin{pmatrix} i, k = 1, \ldots, m \\ l, c, d = 1, \ldots, q;\ c \neq d \end{pmatrix}.$$

6. To derive the result (11′), the only use made of the hypothesis $(k_1, l_1) = (k_2, l_2)$ was to distinguish between the relations (7) and (8). But relations (8), and not (7), are always defined if we take $k_2 \neq k_1$. Hence, if we drop the factor $\frac{1}{2}$ throughout, the investigation in § 5 leads at once to the result

$$(12) \qquad \begin{Bmatrix} a_{k_1 l_1}^{i\ c} & a_{k_2 l_2}^{i\ c} \\ a_{k_1 l_1}^{i\ d} & a_{k_2 l_2}^{i\ d} \end{Bmatrix} = 0 \quad \begin{pmatrix} i, k_1, k_2 = 1, \ldots, m;\ k_1 \neq k_2 \\ c, d = 1, \ldots, q;\ c \neq d \\ l_1, l_2 = 1, \ldots, q \end{pmatrix}.$$

In virtue of the relations (11) and (12), the relations (8) are all satisfied identically.

7. The coefficients of S being not all zero, we may take

$$a_{k_1 l_1}^{i_1 j_1} \neq 0.$$

Then, by (11) and (12) respectively,

$$a_{k_1 l_1}^{i_1 s} = 0 \quad (s = 1, \ldots, q;\ s \neq j_1),$$

$$a_{k_2 l_2}^{i_1 s} = 0 \quad \begin{pmatrix} k_2 = 1, \ldots, m;\ k_2 \neq k_1 \\ l_2 = 1, \ldots, q \end{pmatrix}.$$

Hence the substitution S affects $q-1$ of the indices as follows :—

$$\xi'_{i_1 s} = \sum_{l \neq l_1}^{l=1 \ldots q} a_{k_1 l}^{i_1 s} \xi_{k_1 l} \quad (s = 1, \ldots, q;\ s \neq j_1).$$

Since the determinant of S is not zero, not all of these coefficients are zero, for example,

$$a_{k_1 l_2}^{i_1 j_2} \neq 0 \quad (j_2 \neq j_1,\ l \neq l_1).$$

Then, by (11), $$a_{k_1 l_2}^{i_1 r} = 0 \quad (r = 1, \ldots, q;\ r \neq j_2).$$

Hence S affects $q-2$ of the indices as follows :—

$$\xi'_{i_1 r} = \sum_{l \neq l_1, l_2}^{l=1 \ldots q} a_{k_1 l}^{i_1 r} \xi_{k_1 l} \quad (r = 1, \ldots, q;\ r \neq j_1, j_2).$$

Not all of these coefficients are zero, for example,

$$a_{k_1 l_3}^{i_1 j_3} \quad (j_3 \neq j_1, j_2;\ l_3 \neq l_1, l_2).$$

Proceeding in like manner, we reach, after $q-1$ steps, an index $\xi_{i_1 j_q}$ which S replaces by

$$a_{k_1 l_q}^{i_1 j_q} \xi_{k_1 l_q}.$$

Besides, we have proven the existence of q coefficients

$$(13) \quad a_{k_1 l_1}^{i_1 j_1},\ a_{k_1 l_2}^{i_1 j_2},\ a_{k_1 l_3}^{i_1 j_3},\ \ldots,\ a_{k_1 l_q}^{i_1 j_q},$$

all different from zero, in which $l_1, l_2, \ldots, l_q$ are all distinct, and $j_1, j_2, \ldots, j_q$ all distinct, and consequently each set a permutation of *the integers* 1, 2, ..., q.

The above process may therefore be repeated, starting with any one of the set (13). We conclude that S affects q of the indices as follows :—

$$\xi'_{i_1 j_t} = a^{i_1\, j_t}_{k_1\, l_t}\, \xi_{k_1 l_t} \quad (t = 1, \ldots, q).$$

8. Since the determinant of S is not zero, we may take

$$a^{i_2 j}_{k_2 l} \neq 0 \quad (i_2, k_2) \neq (i_1, k_1).$$

By the argument of § 7, S affects the q indices $\xi_{i_2 1}, \xi_{i_2 2}, \ldots, \xi_{i_2 q}$ as follows :—

$$\xi'_{i_2 j} = a^{i_2 j}_{k_2 l}\, \xi_{k_2 l} \quad (j = 1, \ldots, q).$$

Applying the process m times, we see that S has the form

$$\xi'_{ij} = a^{i\, j}_{k\, l}\, \xi_{k\, l} \quad (i = 1, m\,;\, j = 1, \ldots, q),$$

where k and l are such functions of i and j that, in the determinant of the coefficients of S, no two non-vanishing coefficients lie in the same column or in the same row.

9. *It follows that every linear homogeneous substitution S leaving ϕ invariant is the product of a literal substitution L on the mq letters ξ_{ij} with the systems of imprimitivity*

$$\begin{gathered} \xi_{11},\ \xi_{12},\ \ldots,\ \xi_{1q}, \\ \xi_{21},\ \xi_{22},\ \ldots,\ \xi_{2q}, \\ \xi_{m1},\ \xi_{m2},\ \ldots,\ \xi_{mq}, \end{gathered}$$

by a linear substitution M of the form

$$\xi'_{ij} = a^{ij}_{ij}\, \xi_i \quad (i = 1, m\,;\, j = 1, \ldots, q),$$

where, by (7), $$a^{i1}_{i1}\, a^{i2}_{i2} \ldots a^{iq}_{iq} = 1 \quad (i = 1, \ldots, q).$$

The totality of linear substitutions M form a commutative group which is an invariant sub-group of the total group leaving ϕ invariant. The quotient group is the group of substitutions L. The latter group has an invariant sub-group, the direct product of m symmetric groups on q letters, the quotient group being the symmetric group on m letters, viz., the m systems of imprimitivity.

We have therefore determined completely the structure of the largest linear group leaving invariant the function

$$\phi \equiv \sum_{i=1}^{m} \xi_{i1}\, \xi_{i2} \ldots \xi_{iq},$$

whether the coefficients be taken in the field of continuous quantity, as roots of unity, or, finally, as marks in an arbitrary Galois field.

10. *Note.*—While our final result enables us to give the reciprocal of any substitution S leaving ϕ invariant, it is nevertheless interesting to verify directly by means of the relations (7) and (8) that S^{-1} has the form

$$\xi'_{ij} = \sum_{\substack{k=1\ldots m\\ l=1\ldots q}} A^{kl}_{ij}\,\xi_{kl} \quad (i=1,\ldots,m\,;\ j=1,\ldots,q),$$

where A^{kl}_{ij} denotes the "adjoint" of a^{kl}_{ij} in the symbol

$$\left\{\begin{matrix} a^{k1}_{i1} & a^{k1}_{i2} & \ldots & a^{k1}_{iq} \\ \ldots & \ldots & \ldots & \ldots \\ a^{kq}_{i1} & a^{kq}_{i2} & \ldots & a^{kq}_{iq} \end{matrix}\right\}.$$

For example,
$$A^{k1}_{i2} \equiv \left\{\begin{matrix} a^{k2}_{i1} & a^{k2}_{i3} & \ldots & a^{k2}_{iq} \\ \ldots & \ldots & \ldots & \ldots \\ a^{kq}_{i} & a^{kq}_{i3} & \ldots & a^{kq}_{iq} \end{matrix}\right\}.$$

We verify our statement by showing that $SS^{-1} = 1$. Indeed, SS^{-1} replaces the general index ξ_{ij} by

$$\sum_{\substack{k=1\ldots m\\ l=1\ldots q}} A^{kl}_{ij}\Big(\sum_{\substack{r=1\ldots m\\ s=1\ldots q}} a^{kl}_{rs}\,\xi_{rs}\Big) = \sum_{\substack{r=1\ldots m\\ s=1\ldots q}} \Big\{\sum_{k=1\ldots m}\Big(\sum_{l=1\ldots q} A^{kl}_{ij}\,a^{kl}_{rs}\Big)\Big\}\,\xi_{rs}.$$

But the quantity in brackets is, by (7) and (8),

$$\sum_{k=1\ldots m}\left\{\begin{matrix} a^{k1}_{i1} & \ldots & a^{k1}_{ij-1} & a^{k1}_{rs} & a^{k1}_{ij+1} & \ldots & a^{k1}_{iq} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a^{kq}_{i1} & \ldots & a^{kq}_{ij-1} & a^{kq}_{rs} & a^{kq}_{ij+1} & \ldots & a^{kq}_{iq} \end{matrix}\right\} = \begin{matrix} 0 \text{ if } (r,s) \neq (i,j), \\ 1 \text{ if } (r,s) = (i,j). \end{matrix}$$

Hence SS^{-1} replaces ξ_{ij} by ξ_{ij}.

Groups of Order p^3q. By A. E. WESTERN, M.A.

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1. I propose to discuss the different types of abstract groups whose orders are p^3q, p and q denoting different prime numbers. I must express my thanks to Prof. W. Burnside, F.R.S., for his criticisms on my work, which have enabled me to abbreviate it considerably. Before beginning the consideration of these particular groups it will be well to refer to the previous work of this nature that has been published, and to the general theorems of the theory of groups, of which use will be made.

Throughout this paper the letters $p, q, r, \ldots$ exclusively denote prime numbers, and $\{A, B, \ldots\}$ denotes the group obtained by combining in all possible ways the operations (or groups) $A, B, \ldots$.

There is, as is well known, only one type of group of order p, viz., the cyclic group $\{A\}$, where $A^p = 1$ (Burnside, *Theory of Groups*, p. 26).

There are two types of group of order p^2:—

(i.) $\{A\}$ where $A^{p^2} = 1$ (and no lesser power of A equal to 1; this proviso will in future be implied, for the sake of brevity).

(ii.) $\{A, B\}$, where $A^p = B^p = 1$, $AB = BA$.

Both of these types are Abelian (*alias* "commutative"). (Burnside, *Theory of Groups*, pp. 63 and 81; Young, "On the Determination of Groups whose Order is the Power of a Prime," *Amer. Jour. of Math.*, Vol. xv., 1893, p. 132, and Cole and Glover, in the same volume, p. 192.)

There are also two possible types of order pq:—

(i.) $\{A, B\}$, where $A^p = 1$, $B^q = 1$, $AB = BA$; this may also be written $\{C\}$, where $C^{pq} = 1$.

(ii.) $\{A, B\}$, where $A^p = 1$, $B^q = 1$, and $A^{-1}BA = B^a$, where a is any primitive root of the congruence

$$a^p \equiv 1 \pmod q.$$

This type only exists when $q - 1 \equiv 0 \pmod p$.

(Burnside, *Theory of Groups*, p. 100, and Cole and Glover, *loc. cit.*, pp. 193, 194.)

Groups of order p^3 are dealt with by Burnside, in his *Theory of Groups*, pp. 81, 82, and 87, and by Young, and Cole and Glover, in their papers already referred to. (See also *post*, § 4.)

Groups of order p^2q are given by Burnside, *loc. cit.*, pp. 132–137, and by Cole and Glover, *loc. cit.*

Groups of order pqr are given by Cole and Glover, *loc. cit.*; and, lastly, groups of order p^4 are enumerated by Burnside, *Theory of Groups*, pp. 87, 88, and by Young (*loc. cit.*). See also the memoir by Hölder, "Die Gruppen der Ordnungen p^3, pq^2, pqr, p^4," *Math. Ann.*, Vol. XLIII.

2. Sylow's theorem forms the basis of attack on all groups whose orders contain more than one prime factor. It is expressed by Burnside (p. 92) as follows:—

"If p^a is the highest power of a prime p which divides the order of a group G, the sub-groups of G of order p^a form a single conjugate set, and their number is congruent to unity mod p."

An important corollary is that, if G contains more than one sub-group of order p^a, the order of G must be divisible by $1+kp$ $(k>0)$. For there are, in the case supposed, $1+kp$ sub-groups of order p^a, forming a conjugate set, and the number of sub-groups forming a conjugate set necessarily is a factor of the order of the group.

A second and equally important corollary is that the number of groups of order p^a contained in G can be expressed in the form

$$1+k_1p+k_2p^2+\ldots+k_ap^a,$$

where k_rp^r is the number of groups of order p^a having with a given group H of the set greatest common sub-groups of order p^{a-r} (Burnside, p. 94).

A third, which will also be useful in the sequel, is given by Burnside (p. 94). Using the previous notation, this theorem asserts that, if h is a sub-group common to H and some other sub-group of order p^a such that no sub-group which contains h and is of greater order is common to any two sub-groups of order p^a, then there must be some operation of G of order prime to p which is permutable with h, and not with H.

3. Two other general theorems will be frequently employed later on.

(1) Let G and H be two self-conjugate sub-groups of some third group, having no common operations except identity; then every operation of G is permutable with every operation of H (Burnside, p. 44).

(2) Let $A_1, A_2, \ldots, A_n$ be all the sub-groups (or operations) of a certain type contained in G; and let Q be an operation in G of prime order q. Transform A_1 with respect to Q; the result $Q^{-1}A_1Q$ is a sub-group (or operation) of G of the same type as A_1, and either it is A_1 or it is some other of the set, say A_2. In the latter case, transform A_2 by Q, obtaining A_3, say, and so on, till the cycle closes. Then the cycle contains q of the sub-groups (or operations) $A_1, A_2, \ldots$; for, if possible, suppose the cycle closes with A_x $(x<q)$, so that

$$Q^{-x}A_1Q^x = Q^{-1}A_xQ = A_1.$$

Then we get $$Q^{-xy}A_1Q^{xy} = A_1$$

for all values of y.

Now choose y so that $xy \equiv 1 \pmod{q}$; we thus obtain the result

$$Q^{-1}A_1Q = A_1,$$

which contradicts the hypothesis

$$Q^{-1}A_1Q = A_2.$$

Therefore the sub-groups (or operations) $A_1, A_2, \ldots$ may be divided into l sets of q each, and m each of which is unaltered by transformation with Q, *i.e.*, is permutable with Q; and then

$$n = m + lq.$$

4. The various groups of order p^3 must now be examined, and the facts as to their respective structures proved, which will be needed when I come to consider them as sub-groups of groups of order p^3q. In particular it will be useful to know, as to each group of order p^3, how it may be made isomorphic to itself (see Burnside, chap. xi.); that is, how to find operations $A_0, B_0, \ldots$ in terms of the generating operations $A, B, \ldots$ such that $A_0, B_0, \ldots$ obey the same number of relations, and these of the same form as $A, B, \ldots$; it is obvious that, if this is so, the group may be regarded as generated by $A_0, B_0, \ldots$, just as much as by $A, B, \ldots$.

I. $\{A\}$, where $A^{p^3} = 1$.

This contains one sub-group of order p, $\{A^{p^2}\}$, and one of order p^2, $\{A^p\}$. It is generated by $A_0 = A^x$, provided only that x is prime to p.

This group contains therefore $p^2(p-1)$ operations of order p^3, and so the order of its group of isomorphisms is $p^2(p-1)$. Both its

sub-groups are characteristic sub-groups, *i.e.*, such as are unaltered by every isomorphism of the group (Burnside, p. 232).

II. $\{A, B\}$, where $A^{p^2} = 1$, $B^p = 1$, $AB = BA$.

This contains $p+1$ sub-groups of order p, $\{A^p\}$, and $\{A^{kp}B\}$ (where $k = 0, 1, \ldots, p-1$), and p cyclical sub-groups of order p^2 $\{AB^k\}$ (where $k = 0, 1, \ldots, p-1$), and one non-cyclical sub-group of order p^2 $\{A^p, B\}$.

Let $A_0 = A^xB^y$, $B_0 = A^{zp}B^r$, where x is prime to p, and at least one of z and r is prime to p; then $A_0^{p^2} = 1$, $B_0^p = 1$, and $A_0B_0 = B_0A_0$.

To secure that $\{A_0, B_0\}$ generate the group, we must also ensure that B_0 is independent of A_0.

Suppose that $$B_0 = A_0^k;$$

then $$A^{zp}B^r = A^{xk}B^{yk},$$

and so $$xk \equiv zp \pmod{p^2},$$
$$yk \equiv r \pmod{p}.$$

Since x is prime to p, $k \equiv 0 \pmod{p}$; that is, $r \equiv 0 \pmod{p}$. Provided therefore that $r \not\equiv 0$, A_0 and B_0 generate the group, and are evidently the most general expressions for any possible pair of gènerators. The group of isomorphisms is therefore of order $p^3(p-1)^2$. The characteristic sub-groups are easily seen to be $\{A^p, B\}$ and $\{A^p\}$.

III. $\{A, B, C\}$, where $A^p = B^p = C^p = 1$, $AB = BA$, $AC = CA$, and $BC = CB$.

This contains p^2+p+1 sub-groups of order p, and the same number of order p^2, all of the latter being non-cyclical (Burnside, pp. 59, 60). Every operation of the group is of order p (except 1). $A_0 = A^{a_1}B^{a_2}C^{a_3}$, $B_0 = A^{b_1}B^{b_2}C^{b_3}$, and $C_0 = A^{c_1}B^{c_2}C^{c_3}$ will generate the group, provided that the three congruences given by $A_0^xB_0^yC_0^z = 1$ cannot co-exist; these are

$$a_1x+b_1y+c_1z \equiv 0 \pmod{p},$$
$$a_2x+b_2y+c_2z \equiv 0 \pmod{p},$$
$$a_3x+b_3y+c_3z \equiv 0 \pmod{p}.$$

$a_1a_2 \ldots$ must therefore satisfy the condition

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \not\equiv 0 \pmod{p}.$$

The order of the group of isomorphisms is

$$(p^3-1)(p^3-p)(p^3-p^2)$$

(Burnside, pp. 58, 59); that is,

$$p^3(p-1)^3(p+1)(p^2+p+1).$$

This group evidently has no characteristic sub-groups.

These groups I., II., III. exist whatever prime p may represent, either 2 or any greater prime. Since they are Abelian groups, every sub-group is self-conjugate. The remaining groups of order p^3 differ in form according as p represents 2 or an odd prime; and they are not Abelian groups.

IV. $\{A, B\}$, where $A^4=1$, $B^2=1$, $BAB=A^{-1}$.

The operations of this group are

$$1,\ A,\ A^2,\ A^{-1},\ B,\ AB=BA^{-1},\ A^2B=BA^2,\ A^{-1}B=BA.$$

It contains altogether five sub-groups of order 2: of these one is self-conjugate, $\{A^2\}$; two form a conjugate set, $\{B\}$ and $\{A^2B\}$; two form a conjugate set, $\{AB\}$ and $\{A^{-1}B\}$. And it contains three sub-groups of order 4, all being self-conjugate; two are non-cyclical,

$$\{A^2, B\}=(1, A^2, B, A^2B) \text{ and } \{A^2, AB\}=(1, A^2, AB, A^{-1}B);$$

and one is cyclical, $\{A\}$.

Obviously the most general expressions for A_0 and B_0 are $A_0=A^{\pm 1}$, $B_0=A^xB$ ($x=0$, ± 1, or 2); for then $A_0^4=1$, $B_0^2=1$, and

$$B_0A_0B_0=A^xBA^{\pm1}A^xB=A^xA^{\mp1-x}=A^{\mp1}=A_0^{-1};$$

and evidently A_0 and B_0 are independent, except for the above relations.

The order of the group of isomorphisms is therefore 8.

The characteristic sub-groups are $\{A\}$ and $\{A^2\}$.

V. $\{A, B\}$, where $A^4=1$, $B^2=A^2$, $B^{-1}AB=A^{-1}$.

The operations are $1, A^2, A, A^{-1}, B, A^2B, AB, A^{-1}B$, the latter six being each operations of order 4, and the square of each being A^2.

This group contains one sub-group of order 2, $\{A^2\}$, which is self-conjugate; and three sub-groups of order 4 each cyclical and self-conjugate,

$$\{A\},\ \{AB\}=(1, AB, A^2, A^{-1}B), \text{ and } \{B\}=(1, B, A^2, A^2B).$$

The group is symmetrical in A and B, for from the relations given it follows that $A^{-1}BA=B^{-1}$.

Any independent pair from among the six operations of or[illegible]

may be taken to generate the group, viz.,

$$A_0 = A^{\pm 1}, \quad B_0 = A^k B \quad (k = 0, \ \pm 1, \ 2),$$

or
$$B_0 = A^{\pm 1}, \quad A_0 = A^k B \quad (k = 0, \ \pm 1, \ 2),$$

or
$$A_0 = A^l B, \quad B_0 = A^m B$$
$$(l = 0 \text{ or } 2, \ m = \pm 1, \text{ or } \textit{vice versa}).$$

And in each case $A_0^4 = 1$, $B_0^2 = A_0^2$, and $B_0^{-1} A_0 B_0 = A_0^{-1}$.

The order of the group of isomorphisms is therefore 24. $\{A^2\}$ is the only characteristic sub-group.

VI. $\{A, B\}$, where $A^{p^2} = 1$, $B^p = 1$, $B^{-1}AB = A^{p+1}$, and p is odd.

This contains one self-conjugate sub-group of order p, $\{A^p\}$, and p other sub-groups of order p, $\{A^{kp}B\}$, forming a conjugate set; also p cyclical sub-groups of order p^2, $\{AB^k\}$, which are self-conjugate, and one non-cyclical self-conjugate sub-group of order p^2, $\{A^p, B\}$. In this group

$$(A^a B^b)^x = A^{ax - \frac{1}{2}abpx(x-1)} B^{bx}.$$

Let
$$A_0 = A^a B^b, \quad B_0 = A^{cp} B^d;$$

then $a \not\equiv 0 \pmod p$, or else A_0 would be of order p, and $d \not\equiv 0$, or else B_0 would be a power of A_0. Then

$$\begin{aligned} B_0^{-1} A_0 B_0 &= B^{-d} A^{-cp} A^a B^b A^{cp} B^d \\ &= B^{-d} A^a B^d B^b \\ &= A^{a(1+dp)} B^b \end{aligned}$$

and
$$A_0^{1+p} = A^{a(1+p)} B^b.$$

In order that A_0 and B_0 should take the place of A and B it is necessary that $d = 1$; that is,

$$A_0 = A^a B^b, \quad B_0 = A^{cp} B.$$

And it is easily proved that (if a is prime to p) A_0 and B_0 are not connected by any additional relations.

The order of the group of isomorphisms is therefore $p^3 (p-1)$.

The characteristic sub-groups are $\{A^p\}$ and $\{A^p, B\}$.

VII. $\{A, B, C\}$, where $A^p = B^p = C^p = 1$, $AB = BA$, $AC = CA$, and $C^{-1}BC = AB$; whence also $B^{-1}CB = A^{-1}C$; here p must be odd.

From these we derive

$$C^xB^y = A^{-xy}B^yC^x$$

and

$$(A^aB^bC^c)^x = A^{ax-\frac{1}{2}bcx(x-1)}B^{bx}C^{cx}.$$

Therefore every operation of the group is of order p. A and its powers are the only self-conjugate operations.

This group contains p^2+p+1 sub-groups of order p^2; of these one is self-conjugate, $\{A\}$; the remainder consist of $p+1$ sets, each set containing p conjugates; the sets are

$$\{A^kB\} \quad (k = 0, 1, \ldots, p-1),$$

$$\{A^kC\}, \{A^kBC\}, \ldots, \{A^kB^jC\}, \quad \begin{pmatrix} k = 0, 1, \ldots, p-1 \\ j = 0, 1, \ldots, p-1 \end{pmatrix}.$$

And it contains $p+1$ non-cyclical self-conjugate sub-groups

$$\{A, B\} \text{ and } \{A, B\ C\} \quad (j = 0, 1, \ldots, p-1).$$

The most general transformation of the group into itself that is possible, having regard to $\{A\}$ being the sole self-conjugate sub-group of order p, is

$$A_0 = A^x, \quad B_0 = A^{b_1}B^{b_2}C^{b_3}, \quad C_0 = A^{c_1}B^{c_2}C^{c_3}.$$

Then

$$A_0B_0 = B_0A_0, \quad A_0C_0 = C_0A_0,$$

and

$$C_0^{-1}B_0C_0 = C^{-c_3}B^{-c_2}A^{-c_1}A^{b_1}B^{b_2}C^{b_3}A^{c_1}B^{c_2}C^{c_3}$$

$$= A^{b_1-b_3c_2+b_2c_3}B^{b_2}C^{b_3}$$

and

$$A_0B_0 = A^{b_1+x}B^{b_2}C^{b_3};$$

therefore

$$x \equiv b_2c_3 - b_3c_2 \pmod p.$$

Also the sufficient condition that C_0 should not be expressible in terms of A_0 and B_0 is that $b_2c_3 \not\equiv b_3c_2$, which is, of course, satisfied when the above congruence is satisfied.

To determine the order of the group of isomorphisms, we must find the number of solutions of the congruence

$$x \equiv b_2c_3 - b_3c_2 \pmod p$$

such that x is prime to p.

There are $2p-1$ pairs of values which b_2 and c_3 can assume such that

$$b_2c_3 \equiv 0 \pmod p;$$

with each of these b_3 and c_2 can each take any of the values $1, 2, \ldots, p-1$: thus, if

$$b_2c_3 \equiv 0 \pmod p,$$

there are $(p-1)^2(2p-1)$ solutions; if

$$b_3c_2 \equiv 0 \pmod{p},$$

there are again $(p-1)^2(2p-1)$ solutions.

Lastly, if none of b_2, b_3, c_2, c_3 are congruent to zero, to each of the $(p-1)^3$ sets of values of b_2, b_3, and c_2 there correspond one value of c_3 which makes

$$b_2c_3-b_3c_2 \equiv 0 \pmod{p}$$

and $p-2$ values which do not; in this case then there are $(p-1)^3(p-2)$ solutions.

The order of the group of isomorphisms is therefore

$$p^2\left[2(p-1)^2(2p-1)+(p-1)^3(p-2)\right] = p^3(p-1)^2(p+1).$$

$\{A\}$ is the only characteristic sub-group.

In future I shall refer to these groups by their numbers in this list.

5. *Principles of the Classification of Groups of Order p^3q.*

The application of Sylow's theorem to this order shows that there are either 1 or q sub-groups of order p^3 in a group of order p^3q; in the latter case,

$$q \equiv 1 \pmod{p}.$$

Also there are either 1 or p, or p^2, or p^3 sub-groups of order q in such a group; if p such sub-groups, then

$$p \equiv 1 \pmod{q};$$

if p^2 such sub-groups, then

$$p \equiv 1 \text{ or } -1 \pmod{q};$$

if p^3 such sub-groups, then

$$p \equiv 1 \text{ or } p^2+p+1 \equiv 0 \pmod{q}.$$

Thus the groups of order p^3q fall into four principal divisions:—

(1) Those which contain self-conjugate sub-groups of orders p^3 and q.

(2) Those which contain q sub-groups of order p^3, but a self-conjugate sub-group of order q.

(3) Those which contain a self-conjugate sub-group of order p^3, but more than one sub-group of order q.

(4) Those which do not contain self-conjugate sub-groups of order p^3 or q.

In the remainder of this paper G exclusively denotes a group of order p^3q, and H one of its sub-groups of order p^3.

6. (1) Evidently the sub-groups of orders p^3 and q have no common operation except 1; in this case therefore, applying the theorem of § 3 (1), each operation of order q is permutable with each operation of the sub-group of order p^3.

As in § 4, the letters A, B, and C denote the operations of a group of order p^3, while Q denotes an operation of order q.

Thus, when $p = 2$, there are five groups of this kind for all values of q; viz., the direct products of $\{Q\}$ and the groups I., II., III., IV., and V. of order 8.

And, when $p \neq 2$, there are also five groups for all values of p and q; viz., the direct products of $\{Q\}$, and the groups I., II., III., VI., and VII. of order p^3.

7. *Groups containing q Sub-groups of Order p^3 and one Sub-group only of Order q.*

$$q = 1 \pmod p$$

is a necessary condition for the existence of any group of this kind; evidently then q cannot be 2. It will be convenient to consider separately each of the seven groups of order p^3, subdividing each of these cases in accordance with the values of k_1 and k_2 in the formula (§ 2)

$$q = 1+k_1p+k_2p^2+k_3p^3.$$

Let H represent one of the sub-groups of order p^3; all of them, of course, being conjugates in the group of order p^3q, are of the same type. Then k_1p is the number of such sub-groups having with H greatest common sub-groups of order p^2, k_2p^2 is the number of such sub-groups having with H greatest common sub-groups of order p, and k_3p^3 is the number of such sub-groups having no common operations with H.

(i.) There may exist a sub-group h of order p^2 common to H and some other sub-group of order p^3; this must exist if

$$q \not\equiv 1 \pmod{p^2},$$

and it may also exist if $\quad q \equiv 1 \pmod{p^2}$.

Applying the theorem in § 2, we see that Q is permutable with h.

(ii.) No such sub-group of order p^2 may exist, but there may be a sub-group h of order p common to H and H'; then

$$q \equiv 1 \pmod{p^2};$$

this must exist if $\quad q \not\equiv 1 \pmod{p^3}$,

and it may also exist if $\quad q \equiv 1 \pmod{p^3}$.

Q is permutable with h (§ 2).

(iii.) Lastly, the q sub-groups of order p^3 may have no common operations between any two of them; in this case

$$q \equiv 1 \pmod{p^3}.$$

The group of isomorphisms of any group of order q is a cyclical group of order $q-1$; now, since $\{Q\}$ is self-conjugate in G, every operation of H transforms $\{Q\}$ into itself, and therefore corresponds to an isomorphism of $\{Q\}$. If, then, none of the operations of H are permutable with Q, H is simply isomorphic either to the group of isomorphisms of $\{Q\}$ or to a sub-group of the latter; and so in either case H must be cyclical; this only occurs when H is of type I. If some of the operations of H are permutable with Q, they form a self-conjugate sub-group (which is called h above), of H (Burnside, p. 42); then each operation of the factor-group $\frac{H}{h}$ corresponds to an isomorphism of $\{Q\}$, and therefore $\frac{H}{h}$ must be cyclical. This condition will reduce the number of different cases to be considered.

Further, since h is a self-conjugate sub-group of H, and is permutable with Q, it is a self-conjugate sub-group of G. Also, by hypothesis $\{Q\}$ is a self-conjugate sub-group of G, and evidently h and $\{Q\}$ have no common operations; therefore [§ 3 (1)] every operation of h is permutable with Q.

8. I. $A^{p^3} = 1$.

(i.) h must here be $\{A^p\}$, this being the only sub-group of order p^2 in H. Therefore Q and A^p are permutable operations (§ 7).

And since $\{Q\}$ is self-conjugate, but Q is not permutable with A (a case comprised in § 6),

$$A^{-1}QA = Q^a,$$

where $$a \neq 1.$$

Then $$A^{-p}QA^p = Q^{a^p},$$

and so $$a^p \equiv 1 \pmod q.$$

This congruence has primitive roots, since

$$q \equiv 1 \pmod p.$$

The same type is obtained whichever root of the congruence is taken; for let

$$b \equiv a^x \pmod q,$$

x being prime to p. Then, if $A_0 = A^x$,

$$A_0^{-1}QA_0 = A^{-x}QA^x = Q^{a^x} = Q^b.$$

Thus we obtain one type,

$$A^{p^3} = 1, \quad Q^q = 1, \quad A^{-1}QA = Q^a,$$

where a is any primitive root of

$$a^p \equiv 1 \pmod q, \quad \text{and} \quad q \equiv 1 \pmod p.$$

(ii.) h must now be $\{A^{p^2}\}$, the only sub-group of order p in H. Then Q is permutable with A^{p^2} (§ 7). And so

$$A^{-1}QA = Q^a,$$

where a is a primitive root of

$$a^{p^2} \equiv 1 \pmod q.$$

And, as above, there is only one type, whichever primitive root is taken,

$$A^{p^3} = 1, \quad Q^q = 1, \quad A^{-1}QA = Q^a,$$

where a is any primitive root of

$$a^{p^2} \equiv 1 \pmod q$$

and where $$q \equiv 1 \pmod{p^2}.$$

(iii.) Here $$A^{-1}QA = Q^a,$$

where a is a primitive root of

$$a^{p^3} \equiv 1 \pmod q,$$

and, as above, there is only one type,

$$A^{p^3} = 1, \quad Q^q = 1, \quad A^{-1}QA = Q^a,$$

where a is any primitive root of

$$a^{p^3} \equiv 1 \pmod{q},$$

and where $$q \equiv 1 \pmod{p^3}.$$

9. II. $A^{p^2} = 1,\ B^p = 1,\ AB = BA.$

(i.) This H has two distinct kinds of sub-group of order p^2, cyclic and non-cyclic (§ 4, II.).

First, let h be a cyclic sub-group of order p^2. We saw in § 4, II., that any operation of H whose order is p^2 might be taken as the generator A.

Without loss therefore of generality, we may take $h = \{A\}$. Then $AQ = QA$ (§ 7).

Also, since $\{Q\}$ is self-conjugate,

$$B^{-1}QB = Q^a,$$

where a is a primitive root of

$$a^p \equiv 1 \pmod{q}.$$

Since B^x will do, in place of B, to generate with A the group H, there is only one type,

$$A^{p^2} = 1, \quad B^p = 1, \quad Q^q = 1, \quad AB = BA, \quad AQ = QA, \quad B^{-1}QB = Q^a,$$

where a is any primitive root of

$$a^p \equiv 1 \pmod{q}, \quad \text{and} \quad q \equiv 1 \pmod{p}.$$

Secondly, let $h = \{A^p, B\}$, the only non-cyclic sub-group of order p^2 in H. Then

$$A^pQ = QA^p, \quad BQ = QB \quad (\S\ 7).$$

Therefore $$A^{-1}QA = Q^a,$$

where $$a \neq 1;$$

but, since $$A^{-p}QA^p = Q,$$

a is a primitive root of

$$a^p \equiv 1 \pmod{q}.$$

As before, there is only one type,

$$A^{p^2} = 1, \quad B^p = 1, \quad Q^q = 1, \quad AB = BA, \quad A^{-1}QA = Q^a, \quad BQ = QB,$$

where a is any primitive root of

$$a^p \equiv 1 \pmod{q}, \quad \text{and} \quad q \equiv 1 \pmod{p}.$$

(ii.) Again referring to § 4, II., there are two distinct kinds of sub-group of order p in H, $\{A^p\}$, which is generated by the p^{th} power of an operation, and $\{A^{kp}B\}$; here $A^{kp}B$ is not the p^{th} power of any operation of H. No generality is lost by putting B for $A^{kp}B$ in the latter case.

First, $h = \{A^p\}$. This is impossible, for $\frac{H}{h}$ is a non-cyclic group (§ 7).

Secondly, $h = \{B\}$. Then

$$BQ = QB \text{ (§ 7)}, \quad \text{and} \quad A^{-1}QA = Q^a,$$

where a is any primitive root of

$$a^{p^2} \equiv 1 \pmod{q}, \quad \text{and} \quad q \equiv 1 \pmod{p^2}.$$

These relations define one type.

10. III. $A^p = B^p = C^p = 1$, $AB = BA$, $AC = CA$, $BC = CB$.

(i.) h is here a non-cyclic sub-group of order p^2; suppose it is generated by

$$A_0 = A^{a_1}B^{a_2}C^{a_3}, \quad \text{and} \quad B_0 = A^{b_1}B^{b_2}C^{b_3}.$$

Since A_0 and B_0 are independent, the congruences

$$\frac{a_1}{b_1} \equiv \frac{a_2}{b_2} \equiv \frac{a_3}{b_3} \pmod{p}$$

cannot both be true.

We can therefore choose c_1, c_2, c_3 so that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \not\equiv 0 \pmod{p};$$

therefore, writing $C_0 = A^{c_1}B^{c_2}C^{c_3}$,

A_0, B_0, C_0, generate the group $\{A, B, C\}$; and we have

$$h = \{A_0, A_0\} \text{ (§ 4, III.)}.$$

The suffixes may now be dropped. Thus we have the type given by the relations of III.,

$$AQ = QA, \quad BQ = QB, \quad \text{and} \quad C^{-1}QC = Q^a,$$

where a is any primitive root of

$$a^p \equiv 1 \pmod{q}, \quad \text{and} \quad q \equiv 1 \pmod{p}.$$

11. IV. $A^4 = 1,\ B^2 = 1,\ BAB = A^{-1}$.

(i.) This group has two different kinds of sub-groups of order 4, the cyclical $\{A\}$ and the non-cyclical $\{A^2, B\}$ and $\{A^2, AB\}$ (§ 4, IV).

Firstly, $h = \{A\}$. Then

$$AQ = QA \text{ (§ 7)}, \quad \text{and} \quad B^{-1}QB = Q^a,$$

so that $$a^2 \equiv 1 \pmod q.$$

But $a \not\equiv 1$; so $a \equiv -1$; and we have the type

$$A^4 = 1,\quad B^2 = 1,\quad BAB = A^{-1},\quad Q = 1,\quad AQ = QA,\quad BQB = Q^{-1}.$$

Secondly, $h = \{A^2, B\}$ or $\{A^2, AB\}$; since A and $B_0 = AB$ generate H, and obey the same relations as A and B, it will be sufficient to consider $h = \{A^2, B\}$. Then we get the type

$$A^4 = 1,\quad B^2 = 1,\quad Q^q = 1,\quad BAB = A^{-1},\quad A^{-1}QA = Q^{-1},\quad BQ = QB.$$

(ii.) H has also two different kinds of sub-group of order 2 (§ 4, IV.), $\{A^2\}$ and $\{A^kB\}$ ($k = 0, \pm 1$, or 2). But h cannot be $\{A^2\}$, for $\dfrac{H}{h}$ would then be non-cyclic. Nor can h be one of the other sub-groups of order 2, for they are not self-conjugate (§ 4, IV.).

12. V. $A^4 = 1,\ B^2 = A^2,\ B^{-1}AB = A^{-1}$.

(i.) Let h be some sub-group of order 4; this group contains three such, $\{A\}$, $\{B\}$, and $\{AB\}$, but without loss of generality we can put

$$A_0 = B \quad \text{or} \quad AB,$$

and thus get $$h = \{A\} \text{ (§ 4, V.)}.$$

Then $$AQ = QA \text{ (§ 7)},$$

and then, since $$B^{-2}QB^2 = A^{-2}QA^2 = Q,$$

$$B^{-1}QB = Q^{-1}.$$

Thus we get the type

$$A^4 = 1,\quad B^2 = A^2,\quad B^{-1}AB = A^{-1},\quad Q^q = 1,\quad AQ = QA,\quad B^{-1}QB = Q^{-1}.$$

The only sub-group of order 2 is $\{A^2\}$, and this cannot be h, for then $\dfrac{H}{h}$ would be non-cyclic.

13. VI. $A^{p^2}=1,\ B^p=1,\ B^{-1}AB=A^{p+1}$, and p is odd.

(i.) h is either one of the cyclical sub-groups $\{AB^k\}$, or it is the non-cyclical sub-group $\{A^p,\ B\}$ (§ 4, VI.).

In the first case we can make

$$A_0=AB^k,\quad B_0=B \quad (\S\ 4,\ \text{VI}),$$

and so, dropping suffixes, $h=\{A\}$. Then

$$AQ=QA \ (\S\ 7), \quad \text{and} \quad B^{-1}QB=Q^a,$$

where $$a^p\equiv 1 \pmod q,$$

and a is a primitive root.

We must now find whether any transformation of the group of order p^3q given by these relations for a particular value of a can make the last relation become

$$B_0^{-}\ Q_0B_0=Q_0^b,$$

b being some other root of $\quad a^p\equiv 1 \pmod q$.

Q and its powers are the only operations of order q in the group; clearly nothing is gained by putting $Q_0=Q^x$. $A^fB^gQ^h$ is of order p^2 if f is prime to p, but of order p if f is a multiple of p.

Let $$A_0=A^fB^gQ^h,\quad B_0=A^{jp}B^xQ^k;$$

then $$A_0^{p^2}=1,\quad B_0^p=1,\quad B_0^{-1}QB_0=B^{-x}QB^x=Q^{a^x},$$

and
$$\begin{aligned} B_0^{-1}A_0B_0 &= Q^{-k}B^{-x}A^{-jp}A^fB^gQ^hA^{jp}B^xQ^k \\ &= Q^{-k}B^{-x}A^fB^gQ^hB^xQ^k \\ &= A^{f(1+xp)}Q^{-k}B^{-x+g}Q^hB^xQ^k \\ &= A^{f(1+xp)}Q^{-k}B^gQ^{ha^x+k} \\ &= A^{f(1+xp)}B^gQ^{ha^x-k(a^g-1)}; \end{aligned}$$

also $$A_0^{1+p}=A^{f(1+p)}B^gQ^h;$$

therefore $$f(1+xp)\equiv f(1+p) \pmod{p^2};$$

i.e., $$x\equiv 1 \pmod p.$$

This proves that each primitive root of the congruence

$$a^p\equiv 1 \pmod q$$

gives a separate type; there are therefore $p-1$ types, whose

generating relations are

$$A^{p^2}=1,\ \ B^p=1,\ \ Q^q=1,\ \ B^{-1}AB=A^{p+1},\ \ AQ=QA,\ \ B^{-1}QB=Q^a,$$

or $$Q^{a^2},\ \ldots,\ Q^{a^{p-1}},$$

where a is a primitive root of

$$a^p\equiv 1 \pmod q, \quad \text{and} \quad q\equiv 1 \pmod p.$$

Secondly, $h=\{A^p, B\}$. Then

$$A^pQ=QA^p,\quad BQ=QB \quad (\S\ 7),$$

and then $$A^{-1}QA=Q^a,$$

where a is any primitive root of

$$a^p\equiv 1 \quad (\mathrm{mod}\ q).$$

This only gives one type, for we can take $A_0=A^x$, and all the relations are then unaltered, except that a is replaced by a^x; its relations are

$$A^{p^2}=1,\ \ B^p=1,\ \ Q^q=1,\ \ B^{-1}AB=A^{p+1},\ \ A^{-1}QA=Q^a,\ \ BQ=QB,$$

where a is any primitive root of

$$a^p\equiv 1 \pmod q, \quad \text{and} \quad q\equiv 1 \pmod p.$$

(ii.) $\{A^p\}$ is the only self-conjugate sub-group of order p (§ 4, VI.). $h=\{A^p\}$ makes $\frac{H}{h}$ non-cyclic, which is impossible (§ 7), and therefore no type exists in this case.

14. VII. $A^p=B^p=C^p=1,\ AB=BA,\ AC=CA,\ C^{-1}BC=AB$; p is odd.

(i.) $h=\{A, B\}$ or $\{A, C\}$ or $\{A, B^jC\}$ $(j=1, 2, \ldots, p-1)$.

If $$h=\{A, C\},$$

we can put $$A_0=A^{-1},\quad B_0=C,\quad C_0=B,$$

and so $$h=\{A_0, B_0\}.$$

If $$h=\{A, B^jC\},$$

we can put $$A_0=A^j,\quad B_0=B^jC,\quad C_0=C,$$

and so $$h=\{A_0, B_0\}.$$

It is sufficient then to consider

$$h = \{A, B\}.$$

Then $$AQ = QA, \quad BQ = QB.$$

Also $$C^{-1}QC = Q^{a},$$

where a is a primitive root of $a^p \equiv 1 \pmod q$.

And there is only one type, for, if

$$A_0 = A^x, \quad B_0 = B, \quad C_0 = C^x,$$

thê condition of § 4, VII., is satisfied, and

$$C_0^{-1}QC_0 = Q^{a^x}.$$

The type is

$$A^p = B^p = C^p = Q^q = 1, \quad AB = BA, \quad AC = CA, \quad AQ = QA, \quad BQ = QB,$$
$$C^{-1}BC = AB, \quad C^{-1}QC = Q^a,$$

where a is any primitive root of

$$a^p \equiv 1 \pmod q, \quad \text{and} \quad q \equiv 1 \pmod p.$$

As before, h cannot be of order p, for $\dfrac{H}{h}$ would then be non-cyclic.

15. The third principal division of the subject—groups containing one self-conjugate sub-group of order p^3, but more than one sub-group of order q—must now be considered.

Q, as before, represents any operation of order q in G, and H is the group of order p^3 contained in G. If the operations of H are all transformed by Q, we obtain the same operations in a different order; Q therefore corresponds to an isomorphism of H, and q, the order of Q, must be a divisor of the order of the group of isomorphisms of H. Hence, taking the different types of groups of order p^3 in order (as in § 4), the following congruences involving p and q must hold:—

I. $p \equiv 1 \pmod q$.

II. $p \equiv 1 \pmod q$.

III. $p \equiv 1 \pmod q$, or $p \equiv -1 \pmod q$, or $p^2+p+1 \equiv 0 \pmod q$.

IV. No group exists of the required kind.

V. Here q must divide 24; therefore $q = 3$.

VI. $p \equiv 1 \pmod q$.

VII. $p \equiv 1 \pmod q$ or $p \equiv -1 \pmod q$.

For the same reason Q is permutable with the various characteristic sub-groups of H, named in § 4.

Each of the above cases may be subdivided, according to the number of sub-groups of order q contained in G; this number is either p, p^2, or p^3.

(i.) If G contains p sub-groups of order q, H must contain p^2 operations (forming a sub-group) each of which is permutable with each sub-group of order q; for, if this was not so, the transformation of $\{Q\}$ by each of the operations of H would produce either more or less than p groups of order q. Also, in this case

$$p \equiv 1 \pmod{q} \quad (\S\ 5).$$

(ii.) If G contains p^2 sub-groups of order q, H must (for a similar reason as in the previous case) contain p operations (forming a sub-group) each of which is permutable with each sub-group of order q, and

$$p \equiv 1 \quad \text{or} \quad -1 \pmod{q} \quad (\S\ 5).$$

(iii.) Lastly, if G contains p^3 sub-groups of order q, either

$$p \equiv 1 \pmod{q} \quad \text{or} \quad p^2+p+1 \equiv 0 \pmod{q}.$$

In reference to these congruences it may be noted here that p must be odd when H is either of the types I. and II.; that p must also be odd when G contains p sub-groups of order q; that when $q = 2$ the congruences

$$p \equiv 1 \pmod{q} \quad \text{and} \quad p \equiv -1 \pmod{q}$$

are identical; and that when $q = 3$ the congruences

$$p \equiv 1 \pmod{q} \quad \text{and} \quad p^2+p+1 \equiv 0 \pmod{q}$$

are identical, for $$p^2+p+1 \equiv (p-1)^2 \pmod{3}.$$

Lastly, let D be one of the operations of H mentioned above which are permutable with $\{Q\}$; then, since

$$D^{-1}\{Q\}D = \{Q\}, \quad D^{-1}QD = Q^k,$$

[and so $$D^{-1}(QDQ^{-1}) = Q^{k-1}.$$

Now, H being a self-conjugate sub-group of G, QDQ^{-1} is an operation of H, and therefore $D^{-1}(QDQ^{-1})$, that is, Q^{k-1} is also an operation

of H. Hence
$$Q^{k-1} = 1,$$
and so
$$k = 1.$$
Therefore D and Q are permutable].*

16. I. $A^{p^3} = 1$; $\quad p \equiv 1 \pmod q \quad$ (§ 15).

(i.) p *Sub-groups of Order* q.—The only group of order p^2 here is $\{A^p\}$; so (§ 15)
$$QA^p = A^pQ.$$
Also, since $\{A\}$ is self-conjugate in G,
$$Q^{-1}AQ = A^a;$$
and therefore
$$a^q \equiv 1 \pmod{p^3}.$$
Also
$$Q^{-1}A^pQ = A^{ap} \quad \text{and} \quad Q^{-1}A^pQ = A^p,$$
so
$$a \equiv 1 \pmod{p^2}.$$
Putting
$$a = 1 + kp^2,$$
we get
$$a^q = (1+kp^2)^q \equiv 1 + kqp^2 \pmod{p^3},$$
that is
$$k \equiv 0 \pmod p,$$
and so
$$a \equiv 1 \pmod{p^3}.$$
This makes $AQ = QA$, contrary to hypothesis.

(ii.) p^2 *Sub-groups of Order* q.—Here Q is permutable with A^{p^2} (§ 15), just as in the last case this is inconsistent with
$$Q^{-1}AQ = A^a,$$
a being a primitive root of $\quad a^q \equiv 1 \pmod{p^3}$.

(iii.) p^3 *Sub-groups of Order* q.—Here
$$Q^{-1}AQ = A^a,$$
where a is a primitive root of $\quad a^q \equiv 1 \pmod{p^3}$.

In order that this should have any primitive roots the necessary and sufficient condition is that
$$p \equiv 1 \pmod q.$$

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* Added May 18th, 1899.

By taking $Q_0 = Q^x$, we get a^x in place of a; there is, therefore, when p is odd, one type,

$$A^{p^3} = 1, \quad Q^q = 1, \quad Q^{-1}AQ = A^a,$$

where a is any primitive root of

$$a^q \equiv 1 \pmod{p^3}, \quad \text{and} \quad p \equiv 1 \pmod{q}.$$

17. II. $A^{p^2} = B^p = 1,\ AB = BA,$

$$p \equiv 1 \pmod{q} \quad (\S\ 15).$$

(i.) *p Sub-groups of Order q.*—The group of order p^2 with whose operations Q is permutable (§ 15) is either $\{A^p, B\}$ or $\{AB^k\}$.

First, taking it to be $\{A^p, B\}$, then

$$A^pQ = QA^p, \quad BQ = QB.$$

Then of the p cyclic groups of order p^2 in H one at least [§ 3 (2)] is permutable with Q; if this is $\{AB^k\}$, we can put

$$A_0 = AB^k,$$

and then $\qquad A_0^p = A^p \quad \text{and} \quad Q^{-1}A_0Q = A_0^a.$

Hence $\qquad a^q \equiv 1 \pmod{p^2}, \quad \text{and} \quad ap \equiv p \pmod{p^2}.$

Therefore $\qquad a \equiv 1 \pmod{p^2},$

and G is Abelian, contrary to hypothesis.

Secondly, let Q be permutable with the operations of $\{AB^k\}$; without loss of generality we may write this $\{A\}$. Then $AQ = QA$. Of the p remaining groups of order p in H besides $\{A^p\}$, since

$$p \equiv 1 \pmod{q},$$

one at least [§ 3 (2)] is permutable with Q; without loss of generality, we can take this sub-group to be $\{B\}$, and then

$$Q^{-1}BQ = B^a,$$

where a is any primitive root of

$$a^q \equiv 1 \pmod{p}.$$

Thus there is one type,

$$A^{p^2} = B^p = Q^q = 1, \quad AB = BA, \quad AQ = QA, \quad Q^{-1}BQ = B^a,$$

where a is any primitive root of

$$a^q \equiv 1 \pmod{p}, \quad \text{and} \quad p \equiv 1 \pmod{q}.$$

(ii.) p^2 *Sub-groups of Order* q.—The group of order p with whose operations Q is permutable (§ 15) is either $\{A^p\}$ or $\{A^{kp}B\}$ (of which latter $\{B\}$ may be taken as typical). The case of A^p being permutable with Q may be disposed of just as before.

Next, $BQ = QB$. Of the p cyclic sub-groups of order p^2 one at least is permutable with Q. This may be taken to be $\{A\}$, and then

$$Q^{-1}AQ = A^a,$$

where $$a^q \equiv 1 \pmod{p^2}.$$

Thus we get one type

$$A^{p^2} = B^p = Q^q = 1, \quad AB = BA, \quad Q^{-1}AQ = A^a, \quad BQ = QB,$$

where a is any primitive root of

$$a^q \equiv 1 \pmod{p^2}, \quad \text{and} \quad p \equiv 1 \pmod{q}.$$

(iii.) p^3 *Sub-groups of Order* q.—As before, at least one of the p cyclic sub-groups of order p^2 is permutable with Q, and this may be taken as $\{A\}$, and at least one other besides $\{A^p\}$ of the $p+1$ sub-groups of order p is also permutable with Q; this may be taken as $\{B\}$.

So $$Q^{-1}AQ = A^a,$$

where a is a primitive root of $a^q \equiv 1 \pmod{p^2}$,

and $$Q^{-1}BQ = B^b,$$

where b is a primitive root of $b^q \equiv 1 \pmod{p}$.

How many types do these relations contain? $A^xB^yQ^z$ is of order q, but, so far as its effect in transforming any operation of H is concerned, it is equivalent to Q^z. Putting $Q_0 = Q^z$, we get a^z in place of a, b^z in place of b; a may therefore be fixed as any one of the primitive roots of

$$a^q \equiv 1 \pmod{p^2},$$

and there are $q-1$ types corresponding to the $q-1$ values of b, which may be taken congruent to

$$a, a^2, \ldots, a^{q-1} \pmod{p}.$$

When $$b \not\equiv a \pmod{p},$$

that is, for $q-2$ of these types, none other of the cyclic groups of order p^2 besides $\{A\}$ and none other of the groups of order p besides $\{A^p\}$ and $\{B\}$ are permutable with Q; but, when

$$b \equiv a \pmod{p},$$

all the sub-groups of H are permutable with Q. The relations of these $q-1$ types are

$$A^{p^2} = B^p = Q^q = 1,\quad AB = BA,\quad Q^{-1}AQ = A^a,\quad Q^{-1}BQ = B^a,$$

or $$B^{a^2},\ \ldots,\ \text{or}\ B^{a^{q-1}},$$

where a is any primitive root of

$$a^q \equiv 1 \pmod{p^2},\quad \text{and}\quad p \equiv 1 \pmod{q}.$$

18. III. $A^p = B^p = C^p = 1,\ AB = BA,\ AC = CA,\ BC = CB.$

(i.) *p Sub-groups of Order q;* then

$$p \equiv 1 \pmod{q}.$$

—The group of order p^2 with whose operations Q is permutable (§ 15) may, without loss of generality, be taken to be $\{A, B\}$.

Now H contains p^2+p+1 sub-groups of order p; since

$$AQ = QA,\quad BQ = QB,$$

we know that Q is permutable with $p+1$ of these, viz., $\{A\}$, $\{AB^k\}$. Of the p^2 remaining sub-groups of order p, since

$$p^2 \equiv 1 \pmod{q},$$

there must be at least one other, independent of A and B, which is permutable with Q.

Taking it to be $\{C\}$, we get

$$Q^{-1}CQ = C^a,$$

where a is any primitive root of

$$a^q \equiv 1 \pmod{p},\quad \text{and}\quad p \equiv 1 \pmod{q}.$$

This, combined with the relations of III. and with

$$AQ = QA,\quad BQ = QB,$$

furnishes one type.

19. (ii.) *p^2 Sub-groups of Order q;* and

$$p \equiv 1 \pmod{q}.$$

—The group of order p with whose operations Q is permutable may be taken to be $\{A\}$; then, if $q > 2$, among the p^2+p other sub-groups of order p there are at least two permutable with Q; putting, as we may, $\{B\}$ for one of them, the second may either be $\{A^kB\}$, or else, if

independent of A and B, may be taken as $\{C\}$. But the first of these alternatives is impossible; for

$$AQ = QA, \quad Q^{-1}BQ = B^{\alpha};$$

and therefore $$Q^{-1}A^kBQ = A^kB^a,$$

and this is not a power of A^kB; therefore we must have

$$Q^{-1}CQ = C^b.$$

Here a and b are both primitive roots of

$$a^q \equiv 1 \pmod{p}.$$

We can put $$b \equiv a^x \pmod{p},$$

and the question arises, how many different types are there for different values of x?

So far as altering a and b is concerned, the most general transformation of G is given by

$$Q_0 = Q^y, \quad B_0 = B \text{ or } C, \quad C_0 = C \text{ or } B.$$

Now $$Q_0 = Q^y, \quad B_0 = B, \quad C_0 = C$$

merely amounts to taking a different root of

$$a^q \equiv 1 \pmod{p}$$

for a. On the other hand, if

$$Q_0 = Q^y, \quad B_0 = C, \quad C_0 = B,$$

we get $$Q_0A = AQ_0, \quad Q_0^{-1}B_0Q_0 = B_0^{a^{xy}}, \quad Q_0^{-1}C_0Q_0 = C_0^{a^y}.$$

If, then, we choose y so that $xy \equiv 1 \pmod{q}$,

we have $$a^{xy} \equiv a \pmod{p},$$

and thus we get $Q_0^{-1}B_0Q_0 = B_0^{\alpha}, \quad Q_0^{-1}C_0Q_0 = C_0^{a^y};$

the same relations as before with y in the place of x.

The number of types is therefore the number of solutions of

$$xy \equiv 1 \pmod{q},$$

the order of each pair (x, y) being immaterial.

There are two solutions for which $x \equiv y$, viz.,

$$x \equiv y \equiv 1 \pmod{q}, \quad \text{and} \quad x \equiv y \equiv q-1 \pmod{q}.$$

The remaining $q-3$ residues to the modulus q fall into $\frac{q-3}{2}$ pairs, each pair being a solution of

$$xy \equiv 1 \pmod{q}.$$

Altogether there are $2+\frac{q-3}{2} = \frac{q+1}{2}$ types,

$$A^p = B^p = C^p = Q^q = 1,\ AB = BA,\ AC = CA,\ AQ = QA,\ BC = CB,$$
$$Q^{-1}BQ = B^a,\quad Q^{-1}CQ = C^{a^x},$$

where a is any primitive root of

$$a^q \equiv 1 \pmod{p},$$

x assumes any of the $\frac{q+1}{2}$ values above mentioned, and

$$p \equiv 1 \pmod{q}.$$

[Each of these types is the direct product of $\{A\}$ and $\{B, C, Q\}$].*

The case $q = 2$ was not included above; besides $\{A\}$, either none or at least two groups of order p are permutable with Q; if the latter is the case, we get the one type

$$A^p = B^p = C^p = Q^2 = 1,\ AB = BA,\ AC = CA,\ AQ = QA,$$
$$BC = CB,\ QBQ = B^{-1},\ QCQ = C^{-1}.$$

If, on the other hand, no other group of order p besides $\{A\}$ is permutable with Q, QBQ is either A^xB^y, or, if independent of A and B, may be taken as C; first,

$$QBQ = A^xB^y,$$

where x is not zero. Then

$$B = QA^xB^yQ = A^{x+xy}B^{y^2};$$

and therefore $\qquad y \equiv -1 \pmod{p}.$

But now $\qquad QA^{-x}B^2Q = A^{-x}A^{2x}B^{-2} = (A^{-x}B^2)^{-1};$

the sub-group $\{A^{-x}B^2\}$ is therefore permutable with Q, contrary to hypothesis.

Secondly, let $\qquad QBQ = C,$

then $\qquad QCQ = B;$

and therefore $\qquad Q(BC)Q = BC,$

again contrary to hypothesis.

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* Added May 16th, 1899.

20. (iii.) p^2 *Sub-groups of Order* q;

$$p \equiv -1 \pmod q,$$

where $q \neq 2$ (§ 15).—The group of order p whose operations are permutable with $\{Q\}$ may be taken to be $\{A\}$; then

$$AQ = QA \quad (\S\, 15).$$

No other group of order p can be permutable with Q, for the congruence

$$a^q \equiv 1 \pmod p$$

has no primitive roots. Since

$$p^2+p+1 \equiv 1 \pmod q,$$

at least one of the sub-groups of order p^2 is permutable with Q. First suppose that this is $\{A, B\}$. Then

$$Q^{-1}BQ = A^aB^b;$$

and therefore $$Q^{-x}BQ^x = A^{a(1+b+\ldots+b^{x-1})}B^{b^x};$$

therefore, when $x = q$,

$$B = A^{a(1+b+\ldots+b^{q-1})}B^{b^q};$$

then $$b^q \equiv 1 \pmod p,$$

that is, $$b \equiv 1 \pmod p;$$

and then the index of A is

$$a(1+b+\ldots+b^{q-1}) \equiv qa,$$

an impossible result, since $qa \not\equiv 0 \pmod p$.

The sub-group of order p^2 permutable with Q cannot then contain $\{A\}$; it may therefore be taken to be $\{B, C\}$. Then we get

$$AQ = QA, \quad Q^{-1}BQ = C, \quad Q^{-1}CQ = B^aC^b.$$

$\{B, C, Q\}$ is a group of order p^2q, which is discussed by Burnside in his *Theory of Groups*, p. 136. He shows that the congruence

$$\iota^2 - b\iota - a \equiv 0 \pmod p$$

is obtained, and, on the assumption that its two roots are distinct, proves that they are Galoisian imaginaries, each satisfying

$$\iota^q \equiv 1 \pmod p.$$

It is easy to verify that a and b cannot have such values that this quadratic congruence has equal roots. We thus get one type, the

direct product of $\{A\}$ and $\{B, C, Q\}$, the defining relations of the latter being

$$B^p = C^p = Q^q = 1, \quad BC = CB, \quad Q^{-1}BQ = C, \quad Q^{-1}CQ = B^{-1}C^{\iota^p+\iota},$$

where ι is any primitive (Galoisian) root of the congruence

$$\iota^q \equiv 1 \pmod p, \quad \text{and} \quad p+1 \equiv 0 \pmod q \text{ and } q > 2.$$

21. (iv.) p^3 *Sub-groups of Order* q; and

$$p \equiv 1 \pmod q.$$

—If $q > 3$, since $$p^2+p+1 \equiv 3 \pmod q,$$

at least three groups of order p are permutable with Q; let $\{A\}$ and $\{B\}$ be two of these; then

$$Q^{-1}AQ = A^a, \quad Q^{-1}BQ = B;$$

if a is not equal to b, the third must be independent of A and B, and may be taken as $\{C\}$; if a is equal to b, then $\{A\}$ and $\{A^kB\}$ are $p+1$ groups of order p permutable with Q, and there must therefore be at least one more, $\{C\}$. We therefore get

$$Q^{-1}AQ = A^a, \quad Q^{-1}BQ = B^{a^x}, \quad Q^{-1}CQ = C^{a^y},$$

where a is a primitive root of

$$a^q \equiv 1 \pmod p,$$

and x and y may have any of the values $1, 2, \ldots, q-1$. The somewhat difficult matter remains to determine the number of types comprised in these relations.

As in similar cases before, it suffices to consider the results of taking a power of Q for Q, and permuting the generators of H. In this way we get two distinct equivalences:

First, $$Q_0 = Q^\xi, \quad A_0 = B, \quad B_0 = A, \quad C_0 = C,$$

and $$\xi x \equiv 1 \pmod q;$$

then $$Q_0^{-1}A_0Q_0 = A_0^a, \quad Q_0^{-1}B_0Q_0 = B_0^{a^\xi}, \quad Q_0^{-1}C_0Q_0 = C_0^{a^{\xi y}}.$$

Second, $$Q_0 = Q^\eta, \quad A_0 = C, \quad B_0 = B, \quad C_0 = A,$$

and $$\eta y \equiv 1 \pmod q;$$

then $$Q_0^{-1}A_0Q_0 = A_0^a, \quad Q_0^{-1}B_0Q_0 = B_0^{a^{\eta x}}, \quad Q_0^{-1}C_0Q_0 = C_0^{a^\eta}.$$

Thus, for each pair (x, y), we get corresponding pairs $(\xi, \xi y)$ and $(\eta x, \eta)$; and each of these pairs provides the same type of group; on the other hand, any two pairs (x, y) and (x', y') which are not equivalent correspond to different types. Of course the order of x and y in the symbol (x, y) is immaterial.

It will be convenient to replace these numbers x, y, ξ, &c., by their indices (mod q). Then let

$$x \equiv \gamma^{x_0}, \quad y \equiv \gamma^{y_0}, \quad \xi \equiv \gamma^{-x_0}, \quad \eta \equiv \gamma^{-y_0} \pmod{q};$$

we thus get x_0 and y_0 any two of the complete set of residues to mod $q-1$; viz., 0, 1, 2, ..., $q-2$. And the trio of equivalent pairs is

$$(x_0, y_0), \quad (-y_0, x_0-y_0), \quad (y_0-x_0, -x_0).$$

Let $$\lambda \equiv -y_0, \quad \mu \equiv x_0, \quad \nu \equiv y_0-x_0 \pmod{q-1}.$$

Then $$\lambda+\mu+\nu \equiv 0 \pmod{q-1},$$

and the equivalent pairs are

$$(-\lambda, \mu), \quad (-\mu, \nu), \quad (-\nu, \lambda);$$

and we must now enumerate the solutions of this congruence.

Let α be the number of trios (λ, μ, ν), disregarding order of λ, μ, ν, in which all three numbers are different, β the similar number in which two only are equal, and γ the similar number in which all three are equal.

If $$q \equiv 1 \pmod 3,$$

$$\gamma = 3,$$

for the solutions of this class are

$$\lambda \equiv \mu \equiv \nu \equiv 0, \text{ or } \equiv \frac{q-1}{3}, \text{ or } \equiv \frac{2(q-1)}{3} \pmod{q-1}.$$

If $$q \equiv 2 \pmod 3,$$

$$\gamma = 1,$$

viz., $$\lambda \equiv \mu \equiv \nu \equiv 0 \pmod{q-1}.$$

Next, when two are equal, the congruence is

$$\lambda+2\mu \equiv 0 \pmod{q-1}.$$

μ must not be $\equiv 0$, $\frac{q-1}{3}$, or $\frac{2(q-1)}{3}$, for then it would be $\equiv \lambda$.

With these exceptions μ can have any value, and for each value of μ the congruence gives one value of λ. So, when

$$q \equiv 1 \pmod 3,$$
$$\beta = q-4;$$

when
$$q \equiv 2 \pmod 3,$$
$$\beta = q-2.$$

Now the total number of solutions of all kinds of the congruence, considering the order of each trio, is $(q-1)^2$, for μ and ν can each have any one of $q-1$ values, and the congruence gives a corresponding value of λ to each μ and ν.

Also, in terms of α, β, and γ, the total number of solutions considering the order of each trio, is $6\alpha+3\beta+\gamma$. Therefore

$$6\alpha+3\beta+\gamma = (q-1)^2;$$

then, if
$$q \equiv 1 \pmod 3,$$
$$\alpha = \tfrac{1}{6}(q^2-5q+10),$$

but, if
$$q \equiv 2 \pmod 3,$$
$$\alpha = \tfrac{1}{6}(q^2-5q+6).$$

It is necessary to subdivide these α solutions into those (α_0 in number) in which one of the trio is 0, and the remainder (α_1 in number) in which this is not the case.

Now α_0 is the number of solutions of

$$\lambda+\mu \equiv 0 \pmod{q-1},$$

out of the numbers 1, 2, ..., $q-2$, excluding the solution

$$\lambda \equiv \mu \equiv \frac{q-1}{2};$$

so
$$\alpha_0 = \frac{q-3}{2}.$$

Therefore, when
$$q \equiv 1 \pmod 3,$$
$$\alpha_1 = \tfrac{1}{6}(q^2-8q+19),$$

and, when
$$q \equiv 2 \pmod 3,$$
$$\alpha_1 = \tfrac{1}{6}(q^2-8q+15).$$

Each trio λ, μ, ν in which all are unequal and different from zero

corresponds to one set of equivalent pairs $(-\lambda, \mu)$, $(-\mu, \nu)$, $(-\nu, \lambda)$, and therefore to one type of group; altogether these give α_1 types.

Each trio λ, $\mu \equiv -\lambda$, 0 in which all are unequal corresponds to two distinct sets of equivalent pairs, one being $(-\lambda, -\lambda)$, $(\lambda, 0)$, the other (λ, λ), $(-\lambda, 0)$, and therefore to two types of group, altogether $2\alpha_0$ types.

Each trio λ, μ, μ corresponds to the equivalent pairs $(-\lambda, \mu)$, $(\lambda, -\mu)$, $(-\mu, +\mu)$; the trio $-\lambda$, $-\mu$, $-\mu$ corresponds to the same set; when

$$\mu \equiv \frac{q-1}{2},$$

the trios $(\lambda, \mu, \mu)(-\lambda, -\mu, -\mu)$ form the same solution, but the other trios go in pairs, each pair of trios furnishing one type; thus we get altogether from these trios $\frac{\beta-1}{2}+1$, *i.e.*, $\frac{\beta+1}{2}$, types.

Lastly, when $$q \equiv 1 \pmod 3,$$

there are the two distinct types corresponding to $(0, 0)$, and $\left(\frac{q-1}{3}, -\frac{q-1}{3}\right)$, but, when

$$q \equiv 2 \pmod 3,$$

the single type corresponding to $(0, 0)$.

Adding up these numbers, when

$$q \equiv 1 \pmod 3,$$

the number of types is

$$\frac{q^2-8q+19}{6}+q-3+\frac{q-3}{2}+2 = \frac{q^2+q+4}{6};$$

when $$q \equiv 2 \pmod 3,$$

the number is $$\frac{q^2-8q+15}{6}+q-3+\frac{q-1}{2}+1 = \frac{q^2+q}{6}.$$

The relations for these types are

$$A^p = B^p = C^p = Q^q = 1, \quad AB = BA, \quad AC = CA, \quad BC = CB,$$

$$Q^{-1}AQ = A^a, \quad Q^{-1}BQ = B^{a^x}, \quad Q^{-1}CQ = C^{a^y},$$

where a is any primitive root of

$$a^q \equiv 1 \pmod p, \quad p \equiv 1 \pmod q,$$

and x and y are chosen as above described.

The cases $q = 2$ and 3 have been hitherto excluded; it is, however, easy to see that, if there are three independent groups of order p permutable with Q, all the above work, with the exception of the actual enumeration, applies to these cases.

When $q = 2$, we obtain the single type with the relations

$$Q^{-1}AQ = A^{-1}, \quad Q^{-1}BQ = B^{-1}, \quad Q^{-1}CQ = C^{-1},$$

and, when $q = 3$, the two types

$$Q^{-1}AQ = A^{a}, \quad Q^{-1}BQ = B^{a}, \quad Q^{-1}CQ = C^{a},$$

and $$Q^{-1}AQ = A^{a}, \quad Q^{-1}BQ = B^{a}, \quad Q^{-1}CQ = C^{a^2},$$

where a is any primitive root of

$$a^3 \equiv 1 \pmod{p}, \quad \text{and} \quad p \equiv 1 \pmod{3}.$$

There still remain other possible cases for $q = 2$ or 3, which, however, on examination lead to no fresh types.

$q = 2$.—Suppose that $\{A\}$ is the only group of order p permutable with Q; then

$$QAQ = A^{-1}.$$

Either $$QBQ = A^xB^y,$$

or it may be taken to be C.

In the first case, $$B = A^{-x+xy}B^{y^2},$$

so $$y = 1,$$

and then $$Q(A^xB^2)Q = A^{-x}A^{2x}B^2 = A^xB^2,$$

which is contrary to hypothesis.

Secondly, $$QBQ = C;$$

then $$QCQ = B,$$

and so $$Q(BC)Q = BC,$$

again contrary to hypothesis.

$q = 3$.—Here, since it is supposed that there are not three groups of order p permutable with Q, there are none such; then

$$Q^{-1}AQ = B \text{ (say)}, \quad \text{and} \quad Q^{-1}BQ = A^xB^y \text{ or } C \text{ (say)}.$$

In the first case $$A = Q^{-1}A^xB^yQ = A^{xy}B^{x+y^2},$$

and so $$\left.\begin{array}{l} xy \equiv 1 \\ x \equiv -y^2 \end{array}\right\} \pmod{p}.$$

Either $x \equiv y \equiv -1$, or $x \equiv -a$, $y \equiv -a^2$,

a being a primitive root of $a^3 \equiv 1 \pmod{p}$,

and then either $$Q^{-1}(AB^{-a})\,Q = (AB^{-a})^a,$$

or $$Q^{-1}(AB^{-1})\,Q = (AB^{-1})^a;$$

each of which contradicts the hypothesis.

Lastly, if $$Q^{-1}AQ = B, \quad \text{and} \quad Q^{-1}BQ = C,$$

then $$Q^{-1}CQ = A,$$

and therefore $$Q^{-1}(ABC)\,Q = ABC;$$

this again is impossible.

22. (v.) p^3 *Sub-groups of Order* q, and
$$p^2+p+1 \equiv 0 \pmod{q};$$

then $q > 3$ (§ 15).—None of the groups of order p can be permutable with Q, for, if
$$Q^{-1}AQ = A^a,$$

then $$a^q \equiv 1 \pmod{p};$$

but, q being a divisor of p^2+p+1, must be prime to $p-1$, and therefore
$$a \equiv 1 \pmod{p},$$
which is impossible.

The p^2+p+1 groups of order p must therefore fall into $\dfrac{p^2+p+1}{q}$ sets, each set being cyclically permuted when its groups are transformed by Q.

Then $Q^{-1}AQ$ is not included in $\{A\}$, and may be taken as B; and $Q^{-1}BQ$ is either A^xB^y or may be taken as C. The former case is, however, impossible; for, if so, $\{A, B, Q\}$ is the group of order p^2q already referred to (§ 20), and a necessary condition for its existence is that
$$p+1 \equiv 0 \pmod{q},$$

which is not true here. We therefore obtain
$$Q^{-1}AQ = B, \quad Q^{-1}BQ = C, \quad Q^{-1}CQ = A^aB^\beta C^\gamma.$$

Let $$Q^{-x}CQ^x = A^{a_x}B^{\beta_x}C^{\gamma_x}.$$

Then a_x, β_x, and γ_x must be such that for $x = q$, but for no smaller

value of x, the following congruences are true :—

$$\left.\begin{array}{lll} \alpha_{x-2}\equiv 1, & \alpha_{x-1}\equiv 0, & \alpha_x \not\equiv 0 \\ \beta_{x-2}\equiv 0, & \beta_{x-1}\equiv 1, & \beta_x\equiv 0 \\ \gamma_{x-2}\equiv 0, & \gamma_{x-1}\equiv 0, & \gamma_x\equiv 1 \end{array}\right\} \pmod p.$$

Since
$$\begin{aligned} A^{\alpha_x}B^{\beta_x}C^{\gamma_x} &= Q^{-1}A^{\alpha_{x-1}}B^{\beta_{x-1}}C^{\gamma_{x-1}}Q \\ &= B^{\alpha_{x-1}}C^{\beta_{x-1}}A^{\alpha\gamma_{x-1}}B^{\beta\gamma_{x-1}}C^{\gamma\gamma_{x-1}}, \end{aligned}$$

$\alpha_x, \beta_x, \gamma_x$ are determined by the linear difference-congruences

$$\left.\begin{array}{l} \alpha_x \equiv \alpha\gamma_{x-1} \\ \beta_x \equiv \alpha_{x-1}+\beta\gamma_{x-1} \\ \gamma_x \equiv \gamma\gamma_{x-1}+\beta_{x-1} \end{array}\right\} \pmod p.$$

Hence
$$\gamma_x-\gamma\,\gamma_{x-1}-\beta\,\gamma_{x-2}-\alpha\,\gamma_{x-3}\equiv 0 \pmod p.$$

The solution of this difference-congruence depends on the congruence

$$\lambda^3-\gamma\lambda^2-\beta\lambda-\alpha\equiv 0 \pmod p.$$

First, suppose that the three roots of this are equal, say λ. Then the proper form for γ_x is

$$\gamma_x \equiv (\delta_1+\delta_2 x+\delta_3 x^2)\,\lambda^x,$$

δ_1, &c., being arbitrary constants.

(Throughout this section, all congruences are to be understood as being to the modulus p, unless otherwise expressed.)

In this case
$$\begin{aligned} \gamma &\equiv \Sigma\lambda_1 \equiv 3\lambda, \\ \beta &\equiv -\Sigma\lambda_1\lambda_2 \equiv -3\lambda^2. \end{aligned}$$

Now
$$\begin{aligned} &\gamma_0\equiv 1 \quad (\text{for } \alpha_1\equiv\alpha\gamma_0, \text{ and } \alpha_1\equiv\alpha), \\ &\gamma_1\equiv\gamma\equiv 3\lambda, \\ &\gamma_2\equiv\gamma\gamma_1+\beta_1\equiv\gamma^2+\beta\equiv 6\lambda^2. \end{aligned}$$

If $p=2$, $\lambda\equiv 1$, and we at once obtain $\gamma_2\equiv 1$; this is impossible.

If $p>2$,
$$\begin{aligned} \delta_1 \qquad\qquad &\equiv 1, \\ \delta_2+\ \delta_2+\ \delta_3 &\equiv 3, \\ \delta_1+2\delta_2+4\delta_3 &\equiv 6, \end{aligned}$$

and so
$$\gamma_x = \tfrac{1}{2}(x+1)(x+2)\,\lambda^x.$$

This does not satisfy the conditions

$$\gamma_{q-2} \equiv \tfrac{1}{2}(q-1)\,q\,.\,\lambda^{q-2} \equiv 0,$$
$$\gamma_{q-1} \equiv \tfrac{1}{2}q\,(q+1)\,\lambda^{q-1} \equiv 0,$$

for these congruences are evidently impossible.

Secondly, let two of the three roots of the congruence

$$\lambda^3-\gamma\lambda^2-\beta\lambda-\alpha \equiv 0$$

be congruent; let them be $\lambda_1, \lambda_2, \lambda_2$. Then the proper form for γ_x is

$$\gamma_x \equiv \delta_1\lambda_1^x+(\delta_2+\delta_3 x)\,\lambda_2^x.$$

Then
$$\delta_1 + \delta_2 \equiv 1,$$
$$\delta_1\lambda_1+\delta_2\lambda_2+\delta_3\lambda_2 \equiv \lambda_1+2\lambda_2,$$
$$\delta_1\lambda_1^2+\delta_2\lambda_2^2+2\delta_3\lambda_2^2 \equiv \lambda_1^2+2\lambda_1\lambda_2+3\lambda_2^2,$$

and so
$$\delta_1 \equiv \frac{\lambda_1^2}{(\lambda_1-\lambda_2)^2}, \quad \delta_2 \equiv \frac{\lambda_2^2-2\lambda_1\lambda_2}{(\lambda_1-\lambda_2)^2},$$

and
$$\delta_3 \equiv \frac{-\lambda_2}{\lambda_1-\lambda_2};$$

therefore $\gamma_x \equiv \dfrac{1}{(\lambda_1-\lambda_2)^2}\left\{\lambda^{x+2}-(x+2)\,\lambda_1\lambda_2^{x+1}+(x+1)\,\lambda_2^{x+2}\right\}.$

The conditions $\gamma_{q-2} \equiv 0, \quad \gamma_{q-1} \equiv 0$

give
$$\lambda_1^q-\lambda_2^q \equiv q\lambda_2^{q-1}(\lambda_1-\lambda_2),$$
$$\lambda_1(\lambda_1^q-\lambda_2^q) \equiv q\lambda_2^q(\lambda_1-\lambda_2).$$

These lead to
$$\lambda_1^q \equiv \lambda_2^q \equiv 0,$$

which is not possible.

The three roots of $\lambda^3-\gamma\lambda^2-\beta\lambda-\alpha \equiv 0$

are therefore incongruent; let them be $\lambda_1, \lambda_2, \lambda_3$. Then

$$\gamma_x \equiv \delta_1\lambda_1^x+\delta_2\lambda_2^x+\delta_3\lambda_3^x,$$

and so
$$\delta_1+\delta_2+\delta_3 \equiv 1,$$
$$\lambda_1\delta_1+\lambda_2\delta_2+\lambda_3\delta_3 \equiv \lambda_1+\lambda_2+\lambda_3,$$
$$\lambda_1^2\delta_1+\lambda_2^2\delta_2+\lambda_3^2\delta_3 \equiv \lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2.$$

Let
$$\Delta \equiv \begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{vmatrix} \equiv (\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_3-\lambda_1).$$

Then $$\Delta\delta_1 \equiv -(\lambda_2-\lambda_3)\,\lambda_1^2.$$

Thus we obtain the results

$$\Delta.\alpha_x \equiv -\alpha\Sigma\,(\lambda_2-\lambda_3)\,\lambda_1^{x+1} \equiv -\Sigma\lambda_2\lambda_3\,(\lambda_2-\lambda_3)\,\lambda_1^{x+2},$$

$$\Delta.\beta_x \equiv \quad \Sigma\,(\lambda_2^2-\lambda_3^2)\,\lambda_1^{x+2},$$

$$\Delta.\gamma_x \equiv -\quad \Sigma\,(\lambda_2-\lambda_3)\,\lambda_1^{x+2}.$$

Now, in the light of the three relations between α_x, β_x, γ_x, α_{x-1}, β_{x-1}, γ_{x-1}, only three of the nine conditions above mentioned are independent; we may take as an independent trio

$$\gamma_{q-2} \equiv 0, \quad \gamma_{q-1} \equiv 0, \quad \gamma_q \equiv 1.$$

From the first two of these

$$(\lambda_2-\lambda_3)\,\lambda_1^q \;+(\lambda_3-\lambda_1)\,\lambda^q \;+(\lambda_1-\lambda_2)\,\lambda_3^q \;\equiv 0,$$

$$(\lambda_2-\lambda_3)\,\lambda_1^{q+1}+(\lambda_3-\lambda_1)\,\lambda_2^{q+1}+(\lambda_1-\lambda_2)\,\lambda_3^{q+1} \equiv 0;$$

and therefore

$$(\lambda_3-\lambda_1)\,\lambda_2^q\,(\lambda_1-\lambda_2)+(\lambda_1-\lambda_2)\,\lambda_3^q\,(\lambda_1-\lambda_3) \equiv 0,$$

that is, $$\lambda_2^q \equiv \lambda_3^q.$$

From the symmetry of these congruences,

$$\lambda_1^q \equiv \lambda_2^q \equiv \lambda_3^q.$$

Thirdly, $$-\Sigma\,(\lambda_2-\lambda_3)\,\lambda_1^{q+2} \equiv \Delta.$$

But $$\Sigma\,(\lambda_2-\lambda_3)\,\lambda_1^{q+2} \equiv \lambda_1^q\Sigma\,(\lambda_2-\lambda_3)\,\lambda_1^2 \equiv -\,\Delta\lambda_1^q.$$

Therefore $$\lambda_1^q \equiv \lambda_2^q \equiv \lambda_3^q \equiv 1,$$

and λ_1, λ_2, λ_3 are primitive roots of the congruence

$$\lambda^q \equiv 1 \pmod p.$$

Since q is not a factor of $p-1$ or p^2-1, but is a factor of p^3-1, λ_1, λ_2, and λ_3 are Galoisian imaginaries of the third order, and the congruence

$$\lambda^3-\gamma\lambda^2-\beta\lambda-\alpha \equiv 0 \pmod p$$

is therefore irreducible.

Let λ be any one of the three λ_1, λ_2, λ_3; then λ^p and λ^{p^2} are the other two; for λ, λ^p, λ^{p^2} are necessarily incongruent, and

$$\begin{aligned} \lambda^{3p} &\equiv (\gamma\lambda^2+\beta\lambda+\alpha)^p \\ &\equiv \gamma^p\lambda^{2p}+\beta^p\lambda^p+\alpha \\ &\equiv \gamma\lambda^{2p}+\beta\lambda^p+\alpha, \end{aligned}$$

and so λ^p, and similarly λ^{p^2}, satisfy the congruence. Then

$$\gamma \equiv \lambda+\lambda^p+\lambda^{p^2},$$

$$\beta \equiv -\lambda^{p+1}-\lambda^{p^2+1}-\lambda^{p^2+p},$$

and $$\alpha \equiv \lambda.\lambda^p.\lambda^{p^2} \equiv 1,$$

since $$p^2+p+1 \equiv 0 \pmod q.$$

Each value of λ therefore defines a single group, with the relations

$$Q^{-1}AQ = B, \quad Q^{-1}BQ = C, \quad Q^{-1}CQ = AB^\beta C^\gamma.$$

I shall now prove that there is the same type, whichever primitive root of
$$\lambda^q \equiv 1 \pmod p$$
is taken. Let $$Q_0 = Q^x;$$

then $$B_0 = Q_0^{-1}AQ_0 = A^{\alpha_{x-2}}B^{\beta_{x-2}}C^{\gamma_{x-2}},$$

$$C_0 = Q_0^{-1}B_0Q_0 = A^{\alpha_{2x-2}}B^{\beta_{2x-2}}C^{\gamma_{2x-2}},$$

$$Q_0^{-1}C_0Q_0 = A^{\alpha_{3x-2}}B^{\beta_{3x-2}}C^{\gamma_{3x-2}},$$

and $$A^{\alpha'}B_0C_0^{\gamma'} = A^{\gamma'\alpha_{2x-2}+\beta'\alpha_{x-2}+\alpha'}B^{\gamma'\beta_{2x-2}+\beta'\beta_{x-2}}C^{\gamma'\gamma_{2x-2}+\beta'\gamma_{x-2}}.$$

Therefore $$Q_0^{-1}C_0Q_0 = A^{\alpha'}B_0^{\beta'}C_0^{\gamma'},$$

provided that α', β', and γ' satisfy the congruences

$$\begin{aligned} \alpha_{3x-2}-\gamma'\alpha_{2x-2}-\beta'\alpha_{x-2}-\alpha' &\equiv 0, \\ \beta_{3x-2}-\gamma'\beta_{2x-2}-\beta'\beta_{x-2} &\equiv 0, \\ \gamma_{3x-2}-\gamma'\gamma_{2x-2}-\beta'\gamma_{x-2} &\equiv 0. \end{aligned}$$

Reverting to the notation λ_1, λ_2, and λ_3 for the roots of the congruence

$$\lambda^3-\gamma\lambda^2-\beta\lambda-\alpha \equiv 0,$$

it is easily seen that these congruences are satisfied by

$$\gamma' \equiv \lambda_1^x+\lambda_2^x+\lambda_3^x,$$

$$\beta' \equiv -\lambda_2^x\lambda_3^x-\lambda_3^x\lambda_1^x-\lambda_1^x\lambda_2^x,$$

$$\alpha' \equiv \lambda_1^x\lambda_2^x\lambda_3^x \equiv 1.$$

For, if α', β', and γ' have these values, we have

$$\lambda_k^{3x}-\gamma'\lambda_k^{2x}-\beta'\lambda_k^x-\alpha' \equiv 0,$$

identically, for $k = 1, 2, 3$; and then

$$\Delta\,(\alpha_{3x-2}-\gamma'\alpha_{2x-2}-\beta'\alpha_{x-2}-\alpha')$$
$$\equiv -\Sigma\,\lambda_2\lambda_3\,(\lambda_2-\lambda_3)(\lambda_1^{3x}-\gamma'\lambda_1^{2x}-\beta'\lambda_1^x-\alpha') \equiv 0,$$

$$\Delta\,(\beta_{3x-2}-\gamma'\beta_{2x-2}-\beta'\beta_{x-2}) \equiv \Sigma\,(\lambda_2^2-\lambda_3^2)(\lambda_1^{3x}-\gamma'\lambda_1^{2x}-\beta'\lambda_1^x)$$
$$\equiv \Sigma\,(\lambda_2^2-\lambda_3^2)\,\alpha' \equiv 0,$$

$$\Delta\,(\gamma_{3x-2}-\gamma'\gamma_{2x-2}-\beta'\gamma_{x-2}) \equiv -\Sigma\,(\lambda_2-\lambda_3)(\lambda_1^{3x}-\gamma'\lambda_1^{2x}-\beta'\lambda_1^x)$$
$$\equiv -\Sigma\,(\lambda_2-\lambda_3)\,\alpha' \equiv 0.$$

The effect of making $Q_0 = Q^x$ is therefore to reproduce the original relations, but with λ_1^x, λ_2^x, λ_3^x in place of λ_1, λ_2, λ_3. Thus the one type exists:

$$A^p = B^p = C^p = Q^q = 1,\quad AB = BA,\quad AC = CA,\quad BC = CB,$$

$$Q^{-1}AQ = B,\quad Q^{-1}BQ = C,\quad Q^{-1}CQ = AB^\beta C^\gamma,$$

where β and γ have the values above stated, and

$$p^2+p+1 \equiv 0 \pmod{q}.$$

23. V. $A^4 = 1$, $B^2 = A^2$, $B^{-1}AB = A^{-1}$.

Since $q = 3$, there must be four sub-groups of order q.

Since $\{A^2\}$ is a characteristic sub-group of H (§ 4, V.), A^2 is permutable with Q.

H contains three sub-groups of order 4, $\{A\}$, $\{B\}$, and $\{AB\}$. Q is either permutable with each, or else transforms them cyclically.

The former case is impossible, for, if

$$Q^{-1}AQ = A^a,$$

then $$a = 1,$$

and so A and Q would be permutable.

Secondly, $$Q^{-1}AQ = A^kB,$$
which may be taken as B (§ 4, V.). Then
$$Q^{-1}AQ = B;$$
this gives $$Q^{-1}A^2Q = B^2 = A^2,$$
which is right. Also $Q^{-1}BQ = AB$ or $A^{-1}B$.
Either of these is consistent, for each makes
$$Q^{-2}BQ^2 = A,$$
which is true, since $$Q^3 = 1.$$

There is, however, but one type; for, taking the first,
$$Q^{-1}AQ = B, \quad Q^{-1}BQ = AB,$$
let $$Q_0 = Q^2, \quad B_0 = AB, \quad A_0 = A \quad (\S\ 4, \text{V.});$$
then $Q_0^3 = 1$, $Q_0^{-1}A_0Q_0 = AB = B_0$, and $Q_0^{-1}B_0Q_0 = B = A_0^{-1}B_0$.
This type is
$$A^4 = B^4 = Q^3 = 1,\ B^2 = A^2,\ B^{-1}AB = A^{-1},\ Q^{-1}AQ = B,\ Q^{-1}BQ = AB.$$

24. VI. $A^{p^2} = B^p = 1$, $B^{-1}AB = A^{p+1}$.—p is odd, and
$$p \equiv 1 \pmod q.$$

(i.) *p Sub-groups of Order q.*—The group of order p^2 with whose operations $\{Q\}$, and therefore Q (§ 15), is permutable, is either $\{AB^k\}$ or $\{A^p, B\}$.

First, suppose that Q is permutable with AB^k. Then we can put
$$A_0 = AB^k, \quad B_0 = B \ (\S\ 4, \text{VI.}),$$
and so, dropping suffixes, $$AQ = QA.$$
Then $\{A^p\}$ is a group of order p permutable with Q; there remain p others; since
$$p \equiv 1 \pmod q,$$
at least one of these latter is also permutable with Q, say $\{A^{cp}B\}$. Then we can substitute B for $A^{cp}B$, and thus obtain
$$Q^{-1}BQ = B^a,$$
where a is a primitive root of $a^q \equiv 1 \pmod p$.

These relations, however, are not consistent; for
$$B^{-1}AQ = A^{p+1}B^{-1}Q = A^{p+1}QB^{-a},$$

and $$B^{-1}QA = QB^{-a}A = QA^{ap+1}B^{-a} = A^{ap+1}QB^{-a},$$

and so $$a \equiv 1 \pmod{p},$$

which is contrary to hypothesis.

Secondly, let the group of order p^2 whose operations are permutable with $\{Q\}$ be $\{A^p, B\}$. Then (§ 15)

$$A^pQ = QA^p, \quad BQ = QB.$$

Of the p cyclic groups of order p^2 at least one is permutable with Q, since

$$p \equiv 1 \pmod{q}.$$

It may be taken to be $\{A\}$, without interfering with the result

$$A^pQ = QA^p,$$

above obtained, for $$(AB^k)^p = A^p \quad (\S\,4, \text{VI.}).$$

Then $$Q^{-1}AQ = A^a,$$

where a is a primitive root of $a^q \equiv 1 \pmod{p^2}$.

But $$Q^{-1}A^pQ = A^{ap},$$

and so $$a \equiv 1 \pmod{p},$$

which is inconsistent with a being a primitive root of

$$a^q \equiv 1 \pmod{p^2}.$$

(ii.) *p^2 Sub-groups of Order q.*—Here Q is permutable with the operations of some group of order p. This cannot be $\{A^p\}$, for the same reason that Q in the last case could not be permutable with the operations of $\{A^p, B\}$.

This group of order p may therefore be taken to be $\{B\}$. So

$$BQ = QB.$$

One of the p cyclic groups of order p^2 is permutable with Q; we may take it to be $\{A\}$. Then

$$Q^{-1}AQ = A^a,$$

and a is a primitive root of $a^q \equiv 1 \pmod{p^2}$.

By taking $Q_0 = Q^x$ in place of Q, we get any other root of this congruence in place of a; hence the single type

$$A^{p^2} = B^p = Q^q = 1, \quad B^{-1}AB = A^{p+1}, \quad BQ = QB, \quad Q^{-1}AQ = A^a,$$

where a is any primitive root of

$$a^q \equiv 1 \pmod{p^2}, \quad \text{and} \quad p \equiv 1 \pmod{q}.$$

(iii.) p^3 *Sub-groups of Order* q.—Of the p cyclic groups of order p^2, one, say $\{A\}$, is permutable with Q. Then

$$Q^{-1}AQ = A^a.$$

Besides $\{A^p\}$, at least one other group of order p, say $\{B\}$, is permutable with Q. Then

$$Q^{-1}BQ = B^b.$$

These relations, however, are mutually inconsistent, unless $b = 1$; for

$$B^{-1}AB = A^{p+1},$$

and so $\quad Q^{-1}B^{-1}ABQ = Q^{-1}A^{p+1}Q = A^{a(p+1)}.$

But $\quad BQ = QB^b, \quad Q^{-1}B^{-1} = B^{-b}Q^{-1};$

therefore $\quad A^{a(p+1)} = B^{-b}Q^{-1}AQB^b = B^{-b}A^aB^b = A^{a(bp+1)},$

and so $\quad b \equiv 1 \pmod{p}.$

This makes $\quad BQ = QB,$

which is contrary to hypothesis.

25. VII. $A^p = B^p = C^p = 1,\ AB = BA,\ AC = CA,\ C^{-1}BC = AB$ $(p > 2)$.

Q is permutable with $\{A\}$, the characteristic sub-group of this group.

(i.) p *Sub-groups of Order* q; then

$$p \equiv 1 \pmod{q}.$$

—The operations of some group of order p^2 are permutable with Q (§ 15); it may be assumed to be $\{A, B\}$. Then

$$AQ = QA, \quad BQ = QB,$$

and Q is thus permutable with $p+1$ groups of order p, viz., $\{B\}$ and $\{AB^k\}$ $(k = 0, 1, \ldots, p-1)$; there remain p^2 other such groups; now

$$p^2 \equiv 1 \pmod{q},$$

so at least one of the latter is permutable with Q. Suppose it is $\{A^kB^mC\}$; then we can put

$$A_0 = A, \quad B_0 = B, \quad C_0 = A^kB^mC,$$

and, dropping suffixes,

$$AQ = QA, \quad BQ = QB, \quad \text{and} \quad Q^{-1}CQ = C^a,$$

where a is a primitive root of $a^q \equiv 1 \pmod{p}$.

These relations are, however, inconsistent with

$$C^{-1}BC = AB.$$

For, since $$BQ = QB,$$

$C^{-1}BC$ is permutable with $C^{-1}QC$. Now

$$C^{-1}QC = QC^{-a+1};$$

therefore $ABQC^{-a+1} = QC^{-a+1}AB = AQA^{a-1}BC^{-a+1} = A^aBQC^{-a+1}$;

therefore $$a \equiv 1 \pmod p,$$

which makes Q permutable with C, contrary to hypothesis.

26. (ii.) p^2 *Sub-groups of Order* q; and

$$p \equiv 1 \pmod q.$$

—The operations of some one group of order p are permutable with Q (§ 15).

This case falls into two principal sections according as (1) this group is $\{A\}$, or (2) some other sub-group of H, say $\{B\}$.

(1) $AQ = QA$.

Besides $\{A\}$, there are p^2+p other groups of order p in H; now

$$p^2+p \equiv 2 \pmod q.$$

Except therefore in the case $q = 2$, in which it may be that no other group of order p is permutable with Q (which supposition will be considered later), there are at least two such besides $\{A\}$ permutable with Q, and, of course, this may be the case when $q = 2$. Taking, as we may, $\{B\}$ to be one of these, the other cannot be $\{A^kB\}$, for

$$Q^{-} A^kBQ = A^kB^a,$$

where $$a \neq 1;$$

and A^kB^a is not a power of $\{A^kB\}$.

The third group of order p permutable with Q may therefore be taken as $\{C\}$. Thus we get

$$Q^{-1}BQ = B^a, \quad Q^{-1}CQ = C^b,$$

where a and b are primitive roots of

$$a^q \equiv 1 \pmod p;$$

a and b, however, are not independent, for

$$C^{-1}BC = AB.$$

Transforming this relation with Q, we obtain

$$Q^{-1}C^{-1}BCQ = Q^{-1}ABQ = AB^{a}.$$

Now $$CQ = QC^{b}, \quad Q^{-1}C^{-1} = C^{-b}Q^{-1};$$

and therefore

$$AB^{a} = Q^{-1}C^{-1}BCQ = C^{-b}Q^{-1}BQC^{b} = C^{-b}B^{a}C^{b} = A^{ab}B^{a} \quad (\S\, 4, \text{VII.}).$$

To render the relations consistent it is necessary that

$$ab \equiv 1 \pmod{p},$$

that is, $$b \equiv a^{q-1} \pmod{p}.$$

It will appear on examination that the other relations may be transformed and combined in every possible manner without any inconsistency emerging, provided that the condition

$$b \equiv a^{q-1} \pmod{p}$$

is satisfied.

The relations furnish one type only, for the transformation $Q_0 = Q^x$ changes a into a^x:

$$A^p = B^p = C^p = Q^q = 1, \quad AB = BA, \quad AC = CA, \quad AQ = QA,$$

$$C^{-1}BC = AB, \quad Q^{-1}BQ = B^{a}, \quad Q^{-1}CQ = C^{a^{q-1}},$$

where a is any primitive root of

$$a^q \equiv 1 \pmod{p}, \quad \text{and} \quad p \equiv 1 \pmod{q}.$$

When $q = 2$, there remains the supposed case of the p^2+p groups of order p being all non-permutable with Q. Either

$$QBQ = A^{x}B^{y},$$

or it may be taken to be C.

First, $$QBQ = A^{x}B^{y};$$

then, since $Q^2 = 1$, $$B = A^{x+xy}B^{y^2},$$

so $$y \equiv -1 \pmod{p}.$$

and then $$Q(A^{-x}B^2)Q = A^{-x}A^{2x}B^{-2} = (A^{-x}B^2)^{-1},$$

which is contrary to hypothesis.

Secondly, $$QBQ = C;$$

then $$QCQ = B,$$

and so $$QBC^{-1}Q = CB^{-1} = (BC^{-1})^{-1},$$

again contrary to hypothesis.

(2) Having disposed of the case $AQ = QA$, we must now consider the second case, $BQ = QB$.

As before, there are at least two other groups of order p besides $\{B\}$ permutable with Q; the only conceivable exception being when $q = 2$; this, however, may easily be proved impossible, as in the previous case. And one of these we know is $\{A\}$. Then

$$Q^{-1}AQ = A^a, \quad QB = BQ.$$

The other group of order p permutable with Q may without loss of generality be taken to be $\{C\}$, and so

$$Q^{-1}CQ = C^b;$$

here $$a^q \equiv b^q \equiv 1 \pmod p.$$

Now, since $C^{-1}BC = AB$, $Q^{-1}C^{-1}BCQ = Q^{-1}ABQ = A^aB$.

Now $$CQ = QC^b,$$

and so $$A^aB = C^{-b}Q^{-1}BQC^b = C^{-b}BC^b = A^bB;$$

and therefore $$b \equiv a \pmod p.$$

The other relations give rise to no fresh conditions and no inconsistencies. We therefore get the one type

$$A^p = B^p = C^p = Q^q = 1, \quad AB = BA, \quad AC = CA, \quad C^{-1}BC = AB,$$
$$Q^{-1}AQ = A^a, \quad QB = BQ, \quad Q^{-1}CQ = C^a,$$

where a is any primitive root of

$$a^q \equiv 1 \pmod p, \quad \text{and} \quad p \equiv 1 \pmod q.$$

27. (iii.) p^3 *Sub-groups of Order* q; and

$$p \equiv -1 \pmod q;$$

here $q > 2$.—Since Q is permutable with $\{A\}$, and the congruence

$$a^q \equiv 1 \pmod p$$

has no real primitive roots, Q must be permutable with A. For the same reason, no other group of order p besides $\{A\}$ can be permutable with Q. Then

$$Q^{-1}BQ = A^aB^\beta, \text{ or else } A^aB^\beta C^\gamma.$$

If $$Q^{-1}BQ = A^aB^\beta,$$

then $$Q^{-q}BQ^q = A^{a(1+\beta+\ldots+\beta^{q-1})}B^{\beta^q},$$

so $$\beta^q \equiv 1 \pmod p,$$

that is $$\beta = 1;$$
and so $$B = A^{aq}B,$$
which is impossible. Therefore
$$Q^{-1}BQ = A^{a}B^{\beta}C^{\gamma}.$$
Let $$A_0 = A^{\gamma},\quad B_0 = B,\quad C_0 = A^{a}B^{\beta}C^{\gamma}\quad (\S\,4,\ \text{VII.}).$$
Then $$A_0Q = QA_0,\quad Q^{-1}B_0Q = C_0.$$
Dropping suffixes, we get
$$AQ = QA,\quad Q^{-1}BQ = C,\quad Q^{-1}CQ = A^{a}B^{\beta}C^{\gamma}.$$
Let $$Q^{-x}CQ^{x} = A^{a_x}B^{\beta_x}C^{\gamma_x}.$$
Then
$$\begin{aligned} A^{a_x}B^{\beta_x}C^{\gamma_x} &= Q^{-1}A^{a_{x-1}}B^{\beta_{x-1}}C^{\gamma_{x-1}}Q \\ &= A^{a_{x-1}}C^{\beta_{x-1}}(A^{a}B^{\beta}C^{\gamma})^{\gamma_{x-1}} \\ &= A^{a_{x-1}+a\gamma_{x-1}-\frac{1}{2}\beta\gamma\gamma_{x-1}(\gamma_{x-1}-1)-\beta\beta_{x-1}\gamma_{x-1}}B^{\beta\gamma_{x-1}}C^{\beta_{x-1}+\gamma\gamma_{x-1}}; \end{aligned}$$
and therefore
$$\left.\begin{aligned} a_x-a_{x-1} &\equiv a\gamma_{x-1}-\beta\gamma\frac{\gamma_{x-1}(\gamma_{x-1}-1)}{2}-\beta\beta_{x-1}\gamma_{x-1} \\ \beta_x &\equiv \beta\gamma_{x-1} \\ \gamma_x &\equiv \gamma\gamma_{x-1}+\beta_{x-1} \end{aligned}\right\}\ (\text{mod } p).$$
Then $$\gamma_x-\gamma\gamma_{x-1}-\beta\gamma_{x-2}\equiv 0.$$
If the roots of the congruence
$$\lambda^2-\gamma\lambda-\beta\equiv 0\quad (\text{mod } p)$$
are equal, each being λ, then
$$\gamma\equiv 2\lambda,\quad \beta\equiv -\lambda^2,$$
and so $$\gamma_x\equiv (1+x)\lambda^x.$$
But $$\gamma_{q-1}\equiv 0,$$
and so $$q\lambda^{q-1}\equiv 0\quad (\text{mod } p),$$
which is impossible. Therefore the roots must be unequal, λ_1 and λ_2 say, and then
$$\gamma_x\equiv\frac{\lambda_1^{x+1}-\lambda_2^{x+1}}{\lambda_1-\lambda_2},\quad \beta_x\equiv\beta\frac{\lambda_1^{x}-\lambda_2^{x}}{\lambda_1-\lambda}\quad (\text{mod } p).$$

Now $$\gamma_{q-1} \equiv 0, \quad \gamma_q \equiv 1;$$

therefore $$\lambda_1^q \equiv \lambda_2^q \equiv 1 \pmod p.$$

Also $$\lambda_2 \equiv \lambda_1^p;$$

therefore $$\lambda_1\lambda_2 \equiv \lambda_1^{p+1} \equiv (\lambda^q)^{(p+1)/q} \equiv 1;$$

and so $$\beta \equiv -1.$$

Then $$2(\lambda_1-\lambda_2)^2(a_x-a_{x-1})$$

$$\equiv (2a-\gamma)(\lambda_1^2-1)\lambda_1^{x-1}+(2a-\gamma)(\lambda_2^2-1)\lambda_2^{x-1}+\lambda_1^{2x+1}-\lambda_1^{2x-1}+\lambda_2^{2x+1}-\lambda^{2x-1},$$

and so

$$2(\lambda_1-\lambda_2)^2 a_{x-1} \equiv (2a-\gamma)(\lambda_1^x+\lambda_2^x+\lambda_1^{x-1}+\lambda_2^{x-1}-\gamma-2)+\lambda_1^{2x-1}+\lambda_2^{2x-1}-\gamma.$$

These values of a_x, β_x, and γ_x satisfy the conditions

$$a_{q-1} \equiv a_q \equiv 0, \quad \beta_{q-1} \equiv 1, \quad \beta_q \equiv 0, \quad \gamma_{q-1} \equiv 0, \quad \gamma_q \equiv 1.$$

Thus we obtain the relations

$$AQ = QA, \quad Q^{-1}BQ = C, \quad Q^{-1}CQ = A^a B^{-1} C^\gamma,$$

where $$\gamma \equiv \lambda+\lambda^p,$$

and λ is any primitive root of $\lambda^q \equiv 1 \pmod p$.

These are self-consistent, and the only question remaining is, How many types are included therein?

Let $$A_0 = A^m, \quad B_0 = A^l B^m C^n, \quad C_0 = A^r C$$

(which express an isomorphism of H, § 4, VII.), and

$$Q_0 = Q^k.$$

Then $$Q_0^{-1} B_0 Q_0$$

$$= Q^{-k} A^l B^m C^n Q^k = A^l (A^{a_{k-1}} B^{-\gamma_{k-2}} C^{\gamma_{k-1}})^m (A^{a_k} B^{-\gamma_{k-1}} C^{\gamma_k})^n$$

$$= A^{l+ma_{k-1}+na_k+\frac{1}{2}m(m-1)\gamma_{k-2}\gamma_{k-1}+\frac{1}{2}n(n-1)\gamma_{k-1}\gamma_k+mn\gamma_{k-1}^2}$$
$$\times B^{-m\gamma_{k-2}-n\gamma_{k-1}} C^{m\gamma_{k-1}+n\gamma_k},$$

and this is to be $$C_0 = A^r C.$$

Then $$r \equiv l+f(m, n), \tag{1}$$

the right side being the index of A in $Q_0^{-1}B_0Q_0$. Also

$$m\gamma_{k-2}+n\gamma_{k-1} \equiv 0, \quad m\gamma_{k-1}+n\gamma_k \equiv 1.$$

These last give $$m \equiv \gamma_{k-1}, \quad n \equiv -\gamma_{k-2},$$

since we have $$\gamma_{k-1}^2 - \gamma_k \gamma_{k-2} \equiv 1,$$
identically. Also $$Q_0^{-1} C_0 Q_0 = Q^{-k} A^r C Q^k$$
$$= A^{r+a_k} B^{-\gamma_{k-1}} C^{\gamma_k},$$

and this is equal to $B_0^{-1} C_0^{\delta}$, which is the same as

$$C^{-n} B^{-m} A^{-l+r\delta} C^{\delta} = A^{-l+r\delta-mn} B^{-m} C^{-n+\delta},$$

provided that $$\delta \equiv \gamma_k - \gamma_{k-2}, \tag{2}$$
and $$-l + r\delta - mn \equiv r + a_k. \tag{3}$$

Congruences (1) and (3) can always be satisfied by proper values for l and r. Also

$$\delta \equiv \gamma_k - \gamma_{k-2} \equiv \lambda_1^k + \lambda_2^k,$$

and this shows that the same type is obtained whatever value of λ is taken among the primitive roots of

$$\lambda^q \equiv 1 \pmod p.$$

Thus there is only one type of group of this kind whose generating relations may be taken to be

$$A^p = B^p = C^p = Q^q = 1, \quad AB = BA, \quad AC = CA, \quad C^{-1}BC = AB,$$
$$AQ = QA, \quad Q^{-1}BQ = C, \quad Q^{-1}CQ = B^{-1}C^{\gamma},$$

where $$\gamma \equiv \lambda + \lambda^p,$$

and λ is any primitive root of $\lambda^q \equiv 1 \pmod p$.

28. (iv.) p^3 *Sub-groups of Order* q; then

$$p \equiv 1 \pmod q.$$

—If $q > 2$, at least two of the p^2+p sub-groups of order p, besides $\{A\}$, are permutable with Q, and this may also be the case when $q = 2$. The possible exceptions to this when $q = 2$ will be treated later.

We have $$Q^{-1}AQ = A^a;$$

let $\{B\}$ be another group of order p permutable with Q; then

$$Q^{-1}BQ = B^b.$$

If a is not equal to b, Q cannot be permutable with $\{A^xB\}$, and so the third group of order p may be taken to be $\{C\}$. And if $a = b$, then we have $p+1$ such groups, viz., $\{A\}$, $\{A^xB\}$ permutable with Q;

there remain p^2, one at least of which is also permutable with Q; here again it may be taken to be $\{C\}$. So

$$Q^{-1}AQ = A^a, \quad Q^{-1}BQ = B^{a^x}, \quad Q^{-1}CQ = C^{a^y},$$

where a is any primitive root of

$$a^q \equiv 1 \pmod{p}.$$

Evidently the alteration of a to a^λ, x and y remaining constant, does not make a fresh type. Since

$$C^{-1}BC = AB,$$

$$Q^{-1}C^{-1}BCQ = Q^{-1}ABQ = A^a B^{a^x}.$$

But
$$CQ = QC^{a^y}.$$

So
$$A^a B^{a^x} = C^{-a^y} Q^{-1} BQC^{a^y} = C^{-a^y} B^{a^x} C^{a^y} = A^{a^{x+y}} B^{a^x}.$$

Therefore
$$a^{x+y} \equiv a \pmod{p},$$

that is,
$$x+y \equiv 1 \pmod{q}.$$

If this condition is satisfied, all the relations are consistent.

It remains to find how many types are included in these relations for different values of x and y.

For this purpose, let $B_0 = A^\lambda B^\mu C^\nu$;

we must take for Q_0 the most general form of operation of order q. Since G contains p^3 sub-groups of order q, every operation of the form $A^f B^g C^h Q^k$ is of order q. Since a may be considered fixed, we may put $k = 1$, and, since A is permutable with B and C, we can omit the A^f; thus we have

$$Q_0 = B^g C^h Q.$$

Then, writing
$$b \equiv a^x, \quad c \equiv a^y,$$

we obtain
$$Q_0^{-1} B_0 Q_0 = A^{a\lambda + ah\mu - ag\nu} B^{b\mu} C^{c\nu}.$$

Also
$$B_0^{b_0} = A^{\lambda b_0 - \frac{1}{2}\mu\nu b_0(b_0-1)} B^{b_0\mu} C^{b_0\nu}.$$

Since μ and ν are not both $\equiv 0$, $b_0 \equiv b$ or $c \pmod{p}$.

So the only change that can be made to b and c is to interchange them; thus x, y and y, x give the same type. The number of types is therefore the number of solutions (order being disregarded), of the congruence

$$x+y \equiv 1 \pmod{q}.$$

Neither x nor y can be 0 or 1, for this would make Q permutable with B or C. When $q = 2$, this congruence has no solutions of the proper kind; when $q > 2$, there is one solution

$$x \equiv y \equiv \frac{q+1}{2}$$

for which $$x \equiv y,$$

and $\frac{q-3}{2}$ solutions for which $x \not\equiv y$.

Thus we obtain altogether $\frac{q-1}{2}$ types:

$$A^p = B^p = C^p = Q^q = 1, \quad AB = BA, \quad AC = CA, \quad C^{-1}BC = AB,$$
$$Q^{-1}AQ = A^a, \quad Q^{-1}BQ = B^{a^x}, \quad Q^{-1}CQ = C^{a^{q+1-x}},$$

where a is any primitive root of

$$a^q \equiv 1 \pmod p,$$

q is greater than 2, $$p \equiv 1 \pmod q,$$

and x takes any of the values $2, 3, \ldots, \frac{q+1}{2}$.

The case $q = 2$.—Here either one only or at least three groups of order p are permutable with Q; the case of three permutable with Q has been already discussed, and shown to be impossible for $q = 2$.

Suppose now that only one group of order p is permutable with Q; it must be $\{A\}$, and so

$$QAQ = A^{-1};$$

and then QBQ cannot belong to $\{A, B\}$ (as in § 26), and so may be taken to be C; then

$$QBQ = C, \quad QCQ = B,$$

and so $$QA^{\frac{1}{2}(p-1)}BCQ = A^{\frac{1}{2}(p-1)}BC,$$

which is contrary to hypothesis.

29. We now reach the fourth and last of the principal divisions of the subject (see § 5)—the groups of order p^3q which do not contain self-conjugate sub-groups of orders p^3 or q.

Since there must now be q groups of order p^3,

$$q \equiv 1 \pmod p,$$

and therefore $$p < q.$$

There cannot therefore be p groups of order q, for this requires that

$$p \equiv 1 \pmod{q}.$$

Nor can there be p^3 groups of order q, for then $p^3(q-1)$ operations of the group are of order q, leaving only p^3 other operations; in this case there can only be one group of order p^3. If there are any groups of the kind now sought for, they must therefore contain p^2 groups of order q, with the condition

$$p+1 \equiv 0 \pmod{q}.$$

The only values of p and q satisfying this and the previous condition

$$q \equiv 1 \pmod{p}$$

are
$$p = 2, \quad q = 3.$$

Accordingly, if there are any such groups, they are of order 24. In Burnside's *Theory of Groups* (pp. 101–104), the groups of this order are discussed, and it is unnecessary for me to reproduce this discussion here; it will suffice to give the generating relations of the sole group which has no self-conjugate sub-groups of order 8 or 3,

$$A^4 = B^2 = Q^3 = 1, \quad BAB = A^{-1}, \quad Q^{-1}A^2Q = B, \quad Q^{-1}BQ = A^2B,$$
$$A^{-1}QA = Q^2A^2B.$$

Summary.

30. It will be best to keep distinct the cases $p =$ and >2.

First, *Groups of Order* $8q$.

	Number of Types.
(1) $A^8 = Q^q = 1,\ AQ = QA$	1
This is the cyclic group of order $8q$.	
(2) $A^4 = B^2 = Q^q = 1,\ AB = BA,\ AQ = QA,\ BQ = QB$...	1
(3) $A^2 = B^2 = C^2 = C^q = 1,\ AB = BA,\ AC = CA,$ $BC = CB,\ AQ = QA,\ BQ = QB,\ CQ = QC$	1
These first three groups are Abelian.	
(4) $A^4 = B^2 = Q^q = 1,\ BAB = A^{-1},\ AQ = QA,\ BQ = QB.$	1
(5) $A^4 = B^4 = Q^q = 1,\ B^2 = A^2,\ B^{-1}AB = A^{-1},\ AQ = QA,$ $BQ = QB$	1
(6) $A^8 = Q^q = 1,\ A^{-1}QA = Q^{-1}$	1
(7) $A^4 = B^2 = Q^q = 1,\ AB = BA,\ AQ = QA,\ BQB = Q^{-1}.$	1

Number of Types.

(8) $A^4 = B^2 = Q^q = 1,\ AB = BA,\ A^{-1}QA = Q^{-1},\ BQ = QB.$ 1

(9) $A^2 = B^2 = C^2 = Q^q = 1,\quad AB = BA,\quad AC = CA,$ $BC = CB,\ AQ = QA,\ BQ = QB,\ CQC = Q^{-1}$...... 1

(10) $A^4 = B^2 = Q^q = 1,\ BAB = A^{-1},\ AQ = QA,\ BQB = Q^{-1}.$ 1

(11) $A^4 = B^2 = Q^q = 1,\quad BAB = A^{-1},\quad A^{-1}QA = Q^{-1},$ $BQ = QB$ 1

(12) $A^4 = B^4 = Q^q = 1,\ B^2 = A^2,\ B^{-1}AB = A^{-1},\ AQ = QA,$ $B^{-1}QB = Q^{-1}$ 1

The above twelve groups exist for all values of q (q being supposed a prime number greater than 2).

In addition, when $q \equiv 1 \pmod 4$, there are:—

(13) $A^8 = Q^q = 1,\ A^{-1}QA = Q^a$, where a is any primitive root of $a^4 \equiv 1 \pmod q$ 1

(14) $A^4 = B^2 = Q^q = 1,\ AB = BA,\ A^{-1}QA = Q^a,\ BQ = QB,$ where a has the same meaning as in the previous group 1

Thus, if $q \equiv 1 \pmod 4$, there are fourteen types.

Lastly, if $q \equiv 1 \pmod 8$, in addition to these, there is:—

(15) $A^8 = Q^q = 1,\ A^{-1}QA = Q^a$, where a is any primitive root of $a^8 \equiv 1 \pmod q$ 1

There are, therefore, twelve, fourteen, or fifteen groups of order $8q$ containing a self-conjugate sub-group of order q, according as $q-1$ is a multiple of 2 only, 4 only, or 8.

In addition, for certain values of q, there are groups not containing a self-conjugate sub-group of order q (i.) when $q = 3$:—

(16) The Galoisian ι in this case satisfies $\iota^3 \equiv 1 \pmod 2$. Therefore $\iota^2 + \iota + 1 \equiv 0 \pmod 2$; and so $A^2 = B^2 = C^2 = Q^3 = 1,\quad AB = BA,\quad AC = CA,$ $BC = CB,\ AQ = QA,\ Q^{-1}BQ = C,\ Q^{-1}CQ = BC$... 1

(17) $A^4 = B^4 = Q^3 = 1,\ B^2 = A^2,\ B^{-1}AB = A^{-1},\ Q^{-1}AQ = B,$ $Q^{-1}BQ = AB$ 1

Number of Types.

(18) $A^4 = B^2 = Q^3 = 1, \quad BAB = A^{-1}, \quad Q^{-1}A^2Q = B,$
$Q^{-1}BQ = A^2B, \quad A^{-1}QA = Q^2A^2B$ 1

And (ii.) when $q = 7$:—

(19) The values of β and γ are

$$\beta \equiv -\lambda^3-\lambda^5-\lambda^6, \quad \gamma \equiv \lambda+\lambda^2+\lambda^4,$$

where $\lambda^7 \equiv 1 \pmod 2$. So $\gamma+\beta \equiv 1 \pmod 2$, and we can take $\beta = 1$, $\gamma = 0$; the type is
$A^2 = B^2 = C^2 = Q^7 = 1, \quad AB = BA, \quad AC = CA,$
$BC = CB, \; Q^{-1}AQ = B, \; Q^{-1}BQ = C, \; Q^{-1}CQ = AB$... 1

There are, therefore, altogether fifteen groups of order 24, being the twelve types which exist for all values of q and the three special types just mentioned. And there are thirteen groups of order 56.

My results for the order 24 are confirmed by Burnside's list (pp. 101–104), in which are given the generating relations of the fifteen groups. And the results just given as to groups of order $8q$ are confirmed, so far as the number of types is concerned, by Dr. Miller, in his paper, "The Operation Groups of Order $8p$, p being any Prime Number," *Philosophical Magazine*, Vol. XLII., pp. 195–200.

31. *Groups of Order* p^3q, *where* p *is odd.*

First, those containing self-conjugate sub-groups of orders p^3 and q.

Number of Types.

(1) $A^{p^3} = 1, \; Q^q = 1, \; AQ = QA$ 1

(2) $A^{p^2} = B^p = Q^q = 1, \; AB = BA, \; AQ = QA, \; BQ = QB$ ... 1

(3) $A^p = B^p = C^p = Q^q = 1, \quad AB = BA, \quad AC = CA,$
$BC = CB, \; AQ = QA, \; BQ = QB, \; CQ = QC$ 1

(4) $A^{p^2} = B^p = Q^q = 1, \quad B^{-1}AB = A^{p+1}, \quad AQ = QA,$
$BQ = QB$ 1

(5) $A^p = B^p = C^p = Q^q = 1, \quad AB = BA, \quad AC = CA,$
$C^{-1}BC = AB, \; AQ = QA, \; BQ = QB, \; CQ = QC$ ... 1

Secondly, those containing a self-conjugate sub-group of order q, but not one of order p^3.

Number of Types.

If $q \equiv 1 \pmod p$, there are the following:—

(6) $A^{p^3} = Q^q = 1$, $A^{-1}QA = Q^a$, where a (here and in the next five groups) is any primitive root of $a^p \equiv 1 \pmod q$ 1

(7) $A^{p^2} = B^p = Q^q = 1$, $AB = BA$, $AQ = QA$, $B^{-1}QB = Q^a$ 1

(8) $A^{p^2} = B^p = Q^q = 1$, $AB = BA$, $A^{-1}QA = Q^a$, $BQ = QB$ 1

(9) $A^p = B^p = C^p = Q^q = 1$, $AB = BA$, $AC = CA$, $BC = CB$, $AQ = QA$, $BQ = QB$, $C^{-1}QC = Q^a$ ... 1

(10) $A^{p^2} = B^p = Q^q = 1$, $B^{-1}AB = A^{p+1}$, $AQ = QA$, $B^{-1}QB = Q^b$, where $b = a$, or a^2, ..., or a^{p-1} .... $p-1$

(11) $A^p = B^p = C^p = Q^q = 1$, $AB = BA$, $AC = CA$, $AQ = QA$, $BQ = QB$, $C^{-1}BC = AB$, $C^{-1}QC = Q^a$ 1

And if $q \equiv 1 \pmod{p^2}$, there are, in addition to the above:—

(12) $A^{p^3} = Q^q = 1$, $A^{-1}QA = Q^a$, where a (here and in the next group) is any primitive root of $a^{p^2} \equiv 1 \pmod q$ 1

(13) $A^{p^2} = B^p = Q^q = 1$, $AB = BA$, $A^{-1}QA = Q^a$, $BQ = QB$ 1

And if $q \equiv 1 \pmod{p^3}$, there is, in addition:—

(14) $A^{p^3} = Q^q = 1$, $A^{-1}QA = Q^a$, where a is any primitive root of $a^{p^3} \equiv 1 \pmod q$ 1

Therefore the number of groups of order p^3q containing a self-conjugate sub-group of order q is 5 when $q \not\equiv 1 \pmod p$, $p+9$ when $q \equiv 1 \pmod p$, $p+11$ when $q \equiv 1 \pmod{p^2}$, and $p+12$ when $q \equiv 1 \pmod{p^3}$.

Thirdly, those containing a self-conjugate sub-group of order p^3, but not one of order q.

When $p \equiv 1 \pmod q$, there are the following types:—

[a denotes a primitive root of $a^q \equiv 1 \pmod p$, a_2 of $a^q \equiv 1 \pmod{p^2}$, and a_3 of $a^q \equiv 1 \pmod{p^3}$.]

	Number of Types.
(15) $A^{p^3} = Q^q = 1,\ Q^{-1}AQ = A^{\alpha_3}$	1
(16) $A^{p^2} = B^p = Q^q = 1,\quad AB = BA,\quad AQ = QA,$ $Q^{-1}BQ = B^{\alpha}$	1
(17) $A^{p^2} = B^p = Q^q = 1,\quad AB = BA,\quad Q^{-1}AQ = A^{\alpha_2},$ $BQ = QB$	1
(18) $A^{p^2} = B^p = Q^q = 1,\quad AB = BA,\quad Q^{-1}AQ = A^{\alpha_2},$ $Q^{-1}BQ = B^{\alpha_2}$, or $B^{\alpha_2^2}$, ..., or $B^{\alpha_2^{q-1}}$	$q-1$
(19) $A^p = B^p = C^p = Q^q = 1,\quad AB = BA,\quad AC = CA,$ $BC = CB,\ AQ = QA,\ BQ = QB,\ Q^{-1}CQ = C^{\alpha}$...	1
(20) $q = 2$. $A^p = B^p = C^p = Q^2 = 1,\quad AB = BA,$ $AC = CA,\ BC = CB,\ AQ = QA,\ QBQ = B^{-1},$ $QCQ = C^{-1}$	1
$q > 2$. $A^p = B^p = C^p = Q^q = 1,\quad AB = BA,$ $AC = CA,\ BC = CB,\ AQ = QA,\ Q^{-1}BQ = B^{\alpha},$ $Q^{-1}CQ = C^{\alpha^{\lambda}}$, where λ represents one of $\frac{q+1}{2}$ values (as shown in § 19)	$\frac{q+1}{2}$
(21) $q \equiv 0$ *or* -1 (*mod* 3). $A^p = B^p = C^p = Q^q = 1,$ $AB = BA,\ AC = CA,\ BC = CB,\ Q^{-1}AQ = A^{\alpha},$ $Q^{-1}BQ = B^{\alpha^x},\ Q^{-1}CQ = C^{\alpha^y}$, where x and y have the values shown in § 21	$\frac{q^2+q}{6}$
$q \equiv 1$ (*mod* 3).—The same relations as in the last case	$\frac{q^2+q+4}{6}$
(22) $A^{p^2} = B^p = Q^q = 1,\quad B^{-1}AB = A^{p+1},\quad BQ = QB,$ $Q^{-1}AQ = A^{\alpha_2}$	1
(23) $A^p = B^p = C^p = Q^q = 1,\quad AB = BA,\quad AC = CA,$ $AQ = QA,\quad C^{-1}BC = AB,\quad Q^{-1}BQ = B^{\alpha},$ $Q^{-1}CQ = C^{\alpha^{q-1}}$	1
(24) $A^p = B^p = C^p = Q^q = 1,\quad AB = BA,\quad AC = CA,$ $C^{-1}BC = AB,\quad Q^{-1}AQ = A^{\alpha},\quad QB = BQ,$ $Q^{-1}CQ = C^{\alpha}$	1

Number of Types.

(25) $q > 2$. $A^p = B^p = C^p = Q^q = 1$, $AB = BA$, $AC = CA$, $C^{-1}BC = AB$, $Q^{-1}AQ = A^a$, $Q^{-1}BQ = B^{a^x}$, $Q^{-1}CQ = C^{a^{q+1-x}}$, where $x = 2$, or 3, ..., or $\frac{q+1}{2}$ .... $\frac{q-1}{2}$

When $p \equiv -1 \pmod q$, and $q > 2$, there are:—

(26) $A^p = B^p = C^p = Q^q = 1$, $AB = BA$, $AC = CA$, $BC = CB$, $AQ = QA$, $Q^{-1}BQ = C$, $Q^{-1}CQ = B^{-1}C^{\iota^p+\iota}$, where ι is any primitive Galoisian root of $\iota^q \equiv 1 \pmod p$ 1

(27) $A^p = B^p = C^p = Q^q = 1$, $AB = BA$, $AC = CA$, $C^{-1}BC = AB$, $AQ = QA$, $Q^{-1}BQ = C$, $Q^{-1}CQ = B^{-1}C^{\iota^p+\iota}$ (ι being the same as in the previous type) 1

And, lastly, when $p^2+p+1 \equiv 0 \pmod q$, and $q > 3$, there is the one type:—

(28) $A^p = B^p = C^p = Q^q = 1$, $AB = BA$, $AC = CA$, $BC = CB$, $Q^{-1}AQ = B$, $Q^{-1}BQ = C$, $Q^{-1}CQ = AB^{-\lambda^{-1}-\lambda^{-p}-\lambda^{-p^2}}C^{\lambda+\lambda^p+\lambda^{p^2}}$, where λ is a Galois imaginary of the third order, which is a primitive root of $\lambda^q \equiv 1 \pmod p$ 1

32. Some interesting facts as to the numbers of types of groups of order p^3q can be derived from the foregoing summary.

The most noticeable fact is that (if certain conditions as to the relations between p and q are satisfied) the number of groups of order p^3q increases indefinitely as p or q increases. This is not the case with groups of orders p, p^2, pq, p^3, or p^4; but it is the case with those of order p^2q (see Burnside, *Theory of Groups*, p. 136), where, when

$$p \equiv 1 \pmod q,$$

the number of types is of the form $aq+b$, a and b being constants.

When $$p \equiv 1 \pmod q,$$

the number of groups of order p^3q having a self-conjugate sub-group of order p^3, but not one of order q,

(i.) if $q = 2$, is 10.

(ii.) If $q>2$, and $=3$, or $\equiv -1 \pmod 3$, the number is $\frac{1}{6}(q^2+13q+36)$, that is $\dfrac{(q+4)(q+9)}{6}$.

(iii.) If $q>2$, and $\equiv 1 \pmod 3$, the number is $\frac{1}{6}(q^2+13q+40)$, that is $\dfrac{(q+5)(q+8)}{6}$.

When $$p \equiv -1 \pmod q, \quad \text{and} \quad q > 2,$$

the number of groups of this sort is 2; and when

$$p^2+p+1 \equiv 0 \pmod q, \quad \text{and} \quad q > 3,$$

the number is 1.

Consequently the total number of groups of order $2p^3$ is 15. This enumeration is confirmed by Dr. Miller's paper in the *Quar. Jour. of Math.*, December, 1898 (see pp. 259-263). It is curious that there should be this same number 15 of groups of order p^4, when p is odd (Burnside, *Theory of Groups*, p. 87), and also of order $8q$, where

$$q \equiv 1 \pmod 8$$

(*ante*, § 30).

Also the total number of groups of order $3p^3$ (where p is odd and > 3) is 19, when

$$p \equiv 1 \pmod 3,$$

but 6 only when $$p \equiv -1 \pmod 3.$$

The total number of groups of order $5p^3$ ($p \neq 2$ or 5) is 26, when

$$p \equiv 1 \pmod 5,$$

6 when $$p \equiv -1 \pmod 5,$$

and 5 when $$p \equiv \pm 2 \pmod 5.$$

And the total number of groups of order $7p^3$ ($p \neq 2$ or 7) is 12, when

$$p = 3,$$

35 when $$p \equiv 1 \pmod 7,$$

6 when $$p \equiv 2, 4, \text{ or } 6 \pmod 7,$$

and 5 when ($p > 3$) $$p \equiv 3 \text{ or } 5 \pmod 7.$$

Finally, I give a table showing the number of types of group for all orders of the form p^3q less than 400.

Order.	Factors of Order.	Number of Types.
24	$2^3 . 3$	15
40	$2^3 . 5$	14
54	$3^3 . 2$	15
56	$2^3 . 7$	13
88	$2^3 . 11$	12
104	$2^3 . 13$	14
135	$3^3 . 5$	5
136	$2^3 . 17$	15
152	$2^3 . 19$	12
184	$2^3 . 23$	12
189	$3^3 . 7$	12
232	$2^3 . 29$	14
248	$2^3 . 31$	12
250	$5^3 . 2$	15
296	$2^3 . 37$	14
297	$3^3 . 11$	5
328	$2^3 . 41$	15
344	$2^3 . 43$	12
351	$3^3 . 13$	13
375	$5^3 . 3$	7
376	$2^3 . 47$	12

On the Complete System of Multilinear Differential Covariants of a single Pfaffian Expression, and of a set of Pfaffian Expressions. By J. Brill, M.A.

Received January 31st, 1899. Read February 9th, 1899. Received in revised form April 5th, 1899.

1. An account of the bilinear covariant of a Pfaffian expression is to be found in Forsyth's *Theory of Differential Equations*, Part I., ch. xi. This covariant involves the first set of Pfaffians belonging to the given expression, and is derived from the said expression by

means of a differential operation. As Forsyth points out, a repetition of this method of derivation upon the covariant itself merely produces an expression which vanishes identically. We can, however, by making use, alternately, of algebraical and differential methods of derivation, produce a series of covariants of the given expression which involve the various orders of derived functions associated with it. The theory of the derivation of these successive covariants is dependent upon a method of successive derivation of the Pfaffians and their allied functions, to which I have called attention in a paper recently published in the *Quarterly Journal of Mathematics**.

We shall suppose our given Pfaffian expression to be

$$X_1 dx_1 + X_2 dx_2 + \dots + X_n dx_n.$$

In connexion with it we shall have the first set of Pfaffians

$$[12] = \frac{\partial X_2}{\partial x_1} - \frac{\partial X_1}{\partial x_2}, \qquad [13] = \frac{\partial X_3}{\partial x_1} - \frac{\partial X_1}{\partial x_3}, \text{ \&c.},$$

connected by a set of relations of the form

$$\frac{\partial}{\partial x_1}[23] - \frac{\partial}{\partial x_2}[13] + \frac{\partial}{\partial x_3}[12] = 0.$$

From the first set of Pfaffians we form the first set of allied functions according to the formulæ

$$[0123] = X_1[23] - X_2[13] + X_3[12], \text{ \&c.}$$

The second set of Pfaffians may be derived from these by means of differential operations; for we have

$$\frac{\partial}{\partial x_1}[0234] - \frac{\partial}{\partial x_2}[0134] + \frac{\partial}{\partial x_3}[0124] - \frac{\partial}{\partial x_4}[0123]$$
$$= 2\{[12][34] - [13][24] + [14][23]\} = 2[1234].$$

The members of the second set of Pfaffians are connected by a set of relations of the type

$$\frac{\partial}{\partial x_1}[2345] - \frac{\partial}{\partial x_2}[1345] + \frac{\partial}{\partial x_3}[1245] - \frac{\partial}{\partial x_4}[1235] + \frac{\partial}{\partial x_5}[1234] = 0,$$

as may readily be proved by taking account of their method of derivation from the first set of allied functions.

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* "Suggestions towards the Formation of a General Theory of Systems of Pfaffian Equations," Pt. I., Vol. XXX., pp. 221–242.

For the second set of allied functions we have

$$[012345] = X_1[2345]-X_2[1345]+X_3[1245]-X_4[1235]+X_5[1234],$$

&c.

As a type of the method of derivation of the third set of Pfaffians from these, we have

$$\begin{aligned}
&\frac{\partial}{\partial x_1}[023456]-\frac{\partial}{\partial x_2}[013456]+\frac{\partial}{\partial x_3}[012456]-\frac{\partial}{\partial x_4}[012356] \\
&\qquad +\frac{\partial}{\partial x_5}[012346]-\frac{\partial}{\partial x_6}[012345] \\
&= [12][3456]-[13][2456]+[14][2356]-[15][2346]+[16][2345] \\
&\qquad +[23][1456]-[24][1356]+[25][1346]-[26][1345] \\
&\qquad +[34][1256]-[35][1246]+[36][1245]+[45][1236] \\
&\qquad -[46][1235]+[56][1234] \\
&= 3\{[12][34][56]-[12][35][46]+[12][36][45]-[13][24][56] \\
&\qquad +[13][25][46]-[13][26][45]+[14][23][56] \\
&\qquad -[14][25][36]+[14][26][35]-[15][23][46] \\
&\qquad +[15][24][36]-[15][26][34]+[16][23][45] \\
&\qquad -[16][24][35]+[16][25][34]\} \\
&= 3[123456].
\end{aligned}$$

These methods of successive derivation are clearly general, a fact which could be rigorously demonstrated with the aid of the method of mathematical induction.

The whole set of derived functions connected with our given Pfaffian expression culminates in a single function, which is either a Pfaffian or an allied function according as the number of independent variables is even or odd.

2. Within the continuum symbolized by our system of independent variables, there will be n independent displacements of the point $(x_1, x_2, \ldots, x_n)$. The elements of any other displacement will be expressible linearly in terms of the elements of these. We will suppose the operating symbols $d_1, d_2, \ldots, d_n$ to relate to such a set of displacements, and will write $d_{23\ldots(s+1)}(x_1, x_2, \ldots, x_s)$ as an

abbreviation for the differential determinant

$$\begin{vmatrix} d_2x_1, & d_2x_2, & \ldots, & d_2x_s \\ d_3x_1, & d_3x_2, & \ldots, & d_3x_s \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ d_{s+1}x_1, & d_{s+1}x_2, & \ldots, & d_{s+1}x_s \end{vmatrix},$$

making use of a similar notation to denote all such determinants as can be formed.

We will use the symbol U as an abbreviation for our Pfaffian expression, affixing to it a subscript identical with that affixed to the d's involved in it, *i.e.*, we will write

$$U_r \equiv X_1 d_r x_1 + X_2 d_r x_2 + \ldots + X_n d_r x_n.$$

Now we will suppose our expression to be transformed with the aid of a point transformation defined by the equations

$$x_1' = f_1(x_1, x_2, \ldots, x_n), \qquad x_2' = f_2(x_1, x_2, \ldots, x_n), \text{ \&c.}$$

The transformed expression and its derivatives will be indicated by adding dashes to the symbols that denote the corresponding untransformed quantities.

Then, since $\qquad U_1 = U_1', \quad U_2 = U_2',$

we have $\qquad d_1U_2 - d_2U_1 = d_1U_2' - d_2U_1'.$

This establishes the existence of the first covariant

$$[12]\, d_{12}(x_1, x_2) + [13]\, d_{12}(x_1, x_3) + \ldots + [23]\, d_{12}(x_2, x_3) + \text{\&c.,*}$$

since we have $\qquad d_1d_2 \equiv d_2d_1.$

We will denote this covariant by the symbol U_{12}. We now have

$$U_1 = U_1', \quad U_2 = U_2', \quad U_3 = U_3',$$
$$U_{12} = U_{12}', \; U_{13} = U_{13}', \; U_{23} = U_{23}',$$

and therefore

$$U_1U_{23} - U_2U_{13} + U_3U_{12} = U_1'U_{23}' - U_2'U_{13}' + U_3'U_{12}'.$$

This gives rise to the second covariant, which we will denote by the symbol U_{123}. Written in full it will be of the form

$$[0123]\, d_{123}(x_1, x_2, x_3) + [0124]\, d_{123}(x_1, x_2, x_4) + \ldots$$
$$\ldots + [0234]\, d_{123}(x_2, x_3, x_4) + \text{\&c.,}$$

* The number of terms in this expression is

$$\tfrac{1}{2}n(n-1).$$

It is to be noted that, throughout this paper, the figures within the square brackets are retained in their numerical order. This expression is easily reconciled with that given by Forsyth, by remembering that

$$[21] = -[12], \quad [31] = -[13], \quad \text{\&c.}$$

and will therefore involve the first set of allied functions in a similar manner to that in which the first set of Pfaffians is involved in U_{12}.

To obtain the third covariant we again have recourse to a differential operation. Its existence is demonstrated by the equation

$$d_1 U_{234} - d_2 U_{134} + d_3 U_{124} - d_4 U_{123} = d_1 U'_{234} - d_2 U'_{134} + d_3 U'_{124} - d_4 U'_{123}.$$

We will denote this covariant by the symbol U_{1234}, writing

$$d_1 U_{234} - d_2 U_{134} + d_3 U_{124} - d_4 U_{123} = 2 U_{1234}.$$

We then have

$$U_{1234} = [1234]\, d_{1234}(x_1, x_2, x_3, x_4) + \ldots + [pqrs]\, d_{1234}(x_p, x_q, x_r, x_s) + \&\text{c.}^*$$

We will denote the fourth covariant by the symbol U_{12345}, defining it by the equation

$$U_1 U_{2345} - U_2 U_{1345} + U_3 U_{1245} - U_4 U_{1235} + U_5 U_{1234} = U_{12345}.$$

The fifth covariant will be defined by the equation

$$d_1 U_{23456} - d_2 U_{13456} + d_3 U_{12456} - d_4 U_{12356} + d_5 U_{12346} - d_6 U_{12345} = 3 U_{123456}.$$

Proceeding in this manner we may obtain the whole set of covariants. Their number will be finite, and they will culminate in an expression involving a single differential determinant, viz.,

$$d_{123\ldots n}(x_1, x_2, \ldots, x_n).$$

The coefficient of this determinant will be the final derived function of the original Pfaffian expression.

From these covariants we may readily deduce a set of formulæ expressing the derived functions of the transformed expression in terms of the derived functions of the same order belonging to the original expression. These formulæ will involve the latter quantities linearly. They may be obtained with the aid of the known formula for transforming differential determinants.† As a special example we may instance the formula

$$[pq]' = [12]\frac{\partial(x_1, x_2)}{\partial(x'_p, x'_q)} + [13]\frac{\partial(x_1, x_3)}{\partial(x'_p, x'_q)} + \ldots + [23]\frac{\partial(x_2, x_3)}{\partial(x'_p, x'_q)} + \&\text{c.}\ddagger$$

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* It is readily verified that all differentials of the second order disappear from the result.

† See Donkin: "On a Class of Differential Equations, including those which occur in Dynamical Problems," *Phil. Trans.* 1854, p. 73.

‡ This result is easily obtained by applying the transformation to the left-hand side of the equation

$$U_{12} = U'_{12},$$

and then equating the coefficients of the differential determinants. It is to be noted that, although the number of independent displacements is finite, yet a suitable set of them may be chosen in an indefinite number of ways.

3. In the paper of mine to which I have referred in Article 1, I have developed for the case of several Pfaffian expressions a set of derived functions analogous to the Pfaffians and allied functions belonging to a single expression. I have shown that the vanishing of the various sets of these, taken in a reverse order, is a necessary consequence of a diminution in the number of the functions which constitute the integral system of the set of equations formed by equating the Pfaffian expressions severally to zero. I have not as yet proved the converse theorem.

These new expressions will be involved in the covariants of the system of Pfaffians in a similar manner to that in which the derived functions of a single Pfaffian expression are involved in its covariants. I shall here confine myself to briefly sketching out the method of derivation of these covariants.

Suppose, in the first place, that we have the two Pfaffian expressions

$$U^{(1)} \equiv X_{11}dx_1 + X_{12}dx_2 + \ldots + X_{1n}dx_n,$$
$$U^{(2)} \equiv X_{21}dx_1 + X_{22}dx_2 + \ldots + X_{2n}dx_n.$$

Each of these expressions will have a set of covariants of its own,

viz., $\qquad U^{(1)}_{12},\ U^{(1)}_{123},\ U^{(1)}_{1234},\ \ldots,$

and $\qquad U^{(2)}_{12},\ U^{(2)}_{123},\ U^{(2)}_{1234},\ \ldots.$

In addition to these there will be a set of covariants belonging to the two expressions simultaneously. The first of these is obtained by means of a purely algebraical operation. We have, in fact,

$$U^{(1)}_1 U^{(2)}_2 - U^{(1)}_2 U^{(2)}_1 = (U')^{(1)}_1 (U')^{(2)}_2 - (U')^{(1)}_2 (U')^{(2)}_1.$$

We will write V_{12} for this first covariant, so that we have

$$U^{(1)}_1 U^{(2)}_2 - U^{(1)}_2 U^{(2)}_1 = V_{12}.$$

The second covariant V_{123} is derived from this by means of a differential operation, and is defined by the equation

$$d_1 V_{23} - d_2 V_{13} + d_3 V_{12} = V_{123}.$$

From this two new covariants may be formed by means of algebraical operations. They are

$$U^{(1)}_1 V_{23} - U^{(1)}_2 V_{13} + U^{(1)}_3 V_{12}$$

and $$U^{(2)}_1 V_{23} - U^{(2)}_2 V_{13} + U^{(2)}_3 V_{12}.$$

For convenience we will denote these by the symbols

$$V^{(1)}_{123} \quad \text{and} \quad V^{(2)}_{123}.$$

From these, a new covariant V_{1234} may be formed by means of algebraical operations. It is defined by either of the expressions

$$U_1^{(1)} V_{234}^{(2)} - U_2^{(1)} V_{134}^{(2)} + U_3^{(1)} V_{124}^{(2)} - U_4^{(1)} V_{123}^{(2)}$$

and $$-\left\{ U_1^{(2)} V_{234}^{(1)} - U_2^{(2)} V_{134}^{(1)} + U_3^{(2)} V_{124}^{(1)} - U_4^{(2)} V_{123}^{(1)} \right\}.$$

It may also be written in the form

$$\left\{ U_1^{(1)} U_2^{(2)} - U_2^{(1)} U_1^{(2)} \right\} V_{34} - \left\{ U_1^{(1)} U_3^{(2)} - U_3^{(1)} U_1^{(2)} \right\} V_{24} + \&c.$$

This takes us as far as we can go with the aid of algebraical operations, as another step would produce an evanescent expression. We must, therefore, again have recourse to differential operations, and we deduce a new covariant V_{12345}, defined by the equation

$$d_1 V_{2345} - d_2 V_{1345} + d_3 V_{1245} - d_4 V_{1235} + d_5 V_{1234} = 2 V_{12345}.$$

Treating this covariant in a similar manner to that in which we treated V_{12}, we may form two new covariants, and again a single one, from either of these. We shall then have to introduce differential operations again. Thus we see that the different methods of derivation will repeat themselves in cycles.

4. As a further instance we will take the case of the three Pfaffian expressions

$$U^{(1)} \equiv X_{11} dx_1 + X_{12} dx_2 + \ldots + X_{1n} dx_n,$$

$$U^{(2)} \equiv X_{21} dx_1 + X_{22} dx_2 + \ldots + X_{2n} dx_n,$$

$$U^{(3)} \equiv X_{31} dx_1 + X_{32} dx_2 + \ldots + X_{3n} dx_n.$$

There will, of course, be the sets of invariants belonging to each expression separately, and to each pair combined. In addition, there will be a set belonging to all three combined. The first of this set, W_{123}, is defined by the equation

$$W_{123} = \begin{vmatrix} U_1^{(1)}, & U_2^{(1)}, & U_3^{(1)} \\ U_1^{(2)}, & U_2^{(2)}, & U_3^{(2)} \\ U_1^{(3)}, & U_2^{(3)}, & U_3^{(3)} \end{vmatrix}.$$

The next, W_{1234}, is derived from this by means of a differential operation, thus:

$$W_{1234} = d_1 W_{234} - d_2 W_{134} + d_3 W_{124} - d_4 W_{123}.$$

From this we may form three others,

$$W_{12345}^{(1)}, \quad W_{12345}^{(2)}, \quad W_{12345}^{(3)},$$

which are defined by means of the expressions

$$U_1^{(1)} W_{2345} - U_2^{(1)} W_{1345} + U_3^{(1)} W_{1245} - U_4^{(1)} W_{1235} + U_5^{(1)} W_{1234},$$
$$U_1^{(2)} W_{2345} - U_2^{(2)} W_{1345} + U_3^{(2)} W_{1245} - U_4^{(2)} W_{1235} + U_5^{(2)} W_{1234},$$
$$U_1^{(3)} W_{2345} - U_2^{(3)} W_{1345} + U_3^{(3)} W_{1245} - U_4^{(3)} W_{1235} + U_5^{(3)} W_{1234}.$$

From each of these we may form two covariants, but there will in reality only be three new ones, which may be defined as follows:—

$$W_{123456}^{(23)} = \begin{vmatrix} U_1^{(2)}, & U_2^{(2)} \\ U_1^{(3)}, & U_2^{(3)} \end{vmatrix} W_{3456} - \begin{vmatrix} U_1^{(2)}, & U_3^{(2)} \\ U_1^{(3)}, & U_3^{(3)} \end{vmatrix} W_{2456} + \&c.$$

In the determinants contained in this expression we shall have for subscripts every combination of two out of the numbers 1, 2, 3, 4, 5, 6. For the subscript attached to the W, multiplying any determinant, we have those numbers out of the set 1, 2, 3, 4, 5, 6 which do not occur as subscripts within the corresponding determinant. With regard to the sign of each product, we have first to write down the subscripts of the symbols within the determinant in the order in which they occur, and then to write after them the numbers in the subscript of the corresponding W; then, according as the number of displacements required to bring these numbers into their proper numerical order is even or odd, so will the required sign be plus or minus. In a similar manner, we may write down the expressions for

$$W_{123456}^{(13)} \quad \text{and} \quad W_{123456}^{(12)}.$$

Only one more covariant can be formed from these by algebraical methods. It may be derived from any one of the three, and the expression defining it is of the form

$$\Sigma \begin{vmatrix} U_1^{(1)}, & U_2^{(1)}, & U_3^{(1)} \\ U_1^{(2)}, & U_2^{(2)}, & U_3^{(2)} \\ U_1^{(3)}, & U_2^{(3)}, & U_3^{(3)} \end{vmatrix} W_{4567}.$$

The law for determining the signs of the various products under the Σ will be the same as in the case of the covariants immediately preceding.

We must now again have recourse to a differential method of derivation; and it is evident that, in this case, the different methods of derivation will repeat themselves in triple cycles.

5. The cases we have discussed are sufficient to illustrate the general method of derivation, the progress of which is perfectly

clear. It will be noted that the places at which differential operations occur are those which mark the passing from one group of cases into the next in the case of a set of equations obtained by equating our Pfaffian expressions severally to zero. Further, it will be found that the more general derived functions that I have introduced play a similar part, in reference to these latter covariants, to that which the derived functions of a single expression play in reference to its covariants. It will also be noticed that the transformation theory applies. One of the main difficulties in working out general proofs of propositions in this subject is the extraordinary complication of the notation. The chief desideratum is the invention of a notation to express the general type of derived functions as convenient as that introduced by Cayley to denote the derived functions of a single expression.

Note on a Case of Divisibility of a Function of Two Variables by another Function. By ARTHUR BERRY, Fellow of King's College, Cambridge. Received and read February 9th, 1899.

§ 1.

If $f = 0$, $\phi = 0$ are the equations, expressed in Cartesian coordinates, of two given algebraic curves, which have both simple and multiple points of intersection, and if $\psi = 0$ is the equation of a third curve passing through all these points, and satisfying certain further conditions at the multiple points of intersection, then we have the identity

$$\psi \equiv Af + B\phi$$

where A, B are polynomials in the coordinates x, y. The above mentioned conditions were first stated, and the theorem rigorously proved, by Noether.* A simpler proof of Noether's theorem was soon afterwards published by Halphen.† Dr. F. S. Macaulay has

* "Ueber einen Satz aus der Theorie der algebraischen Functionen," *Mathematische Annalen*, Vol. VI. (1873).

† "Sur une proposition d'Algèbre," *Bulletin de la Société Mathématique de France*, Vol. V. (1877), the article is reproduced in Benoist's French translation of Clebsch's *Vorlesungen über Geometrie*.

recently called my attention to a step in Halphen's argument which appears to require justification. The object of this note is to establish and to generalize the proposition which Halphen tacitly assumes.

§ 2.

If F, f, Φ, ϕ are four polynomials in two variables x, y, connected by the identity

$$Ff = \Phi\phi, \tag{1}$$

then, provided that f and ϕ have no polynomial as a common factor, we know that F must be divisible by ϕ and Φ by f, that is, that F/ϕ and Φ/f are themselves polynomials. But, if F and Φ are no longer polynomials, but infinite series proceeding by ascending positive integral powers of x, y (*i.e.*, ordinary power series), is the corresponding result true? In the first place, the algebraic notion of divisibility requires modification, when we are no longer dealing with polynomials. According to Weierstrass's definition,* an ordinary power series $p_1(x, y)$ is divisible by another $p_2(x, y)$, both of which converge in the neighbourhood of the origin,† if their quotient p_1/p_2 is expansible as a third power series $p_3(x, y)$, also convergent in the neighbourhood of the origin; this includes the ordinary algebraic definition. Now, if p_2 does not vanish at the origin, but is of the form $a_0 + a_1x + b_1y + \dots$ where $a_0 \neq 0$, then it is known that $1/p_2$ is expressible in the form p_3, and therefore so also is p_1/p_2, whatever p_1 be. If, therefore,

$$\phi(0, 0) \neq 0,$$

we have

$$F/\phi = p_3(x, y),$$

and therefore also

$$\Phi/f = F/\phi = p_3(x, y);$$

but in Halphen's argument, f and ϕ both vanish at the origin, $f = 0$ and $\phi = 0$ being two curves which pass through the origin, and have multiple points of any order there; and therefore this

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* "Einige auf die Theorie der analytischen Functionen mehrerer Veränderlichen sich beziehende Sätze," *Werke*, Vol. II., pp. 135–188. First printed in the *Abhandlungen aus der Functionenlehre* (1886). Weierstrass's results have been used by Stickelberger and by Baker in papers on Noether's theorem, *Mathematische Annalen*, Vols. XXX. (1887), XLII. (1893).

† By "neighbourhood of the origin," I mean the aggregate of values of x, y for which $|x|$, $|y|$ are less than certain finite positive quantities δ, δ', however small. I also use "point" as a convenient abbreviation for a pair of values of x, y.

procedure fails. Halphen, however, does not appear to notice this, and infers (without any attempt at proof) that Φ/f is a convergent power series.

It may be noticed incidentally that the question of the divisibility of functions of two or more variables is much more troublesome than in the case of one variable, since in the former case, if two functions are zero of the same order at any point, their quotient is not in general a determinate finite quantity, but is wholly indeterminate. The behaviour at the origin of such a simple function as $(x+y)/(x-y)$ illustrates this.

Halphen's assumption may, however, be justified by methods given by Weierstrass in the paper already quoted.*

$F(x, y)$ being a power series as defined (assumed not divisible by x), $F(0, y)$ is an ordinary power series in y only; let the lowest power of y which occurs be y^m, where m is an integer $\geqslant 0$, so that m is the number of vanishing roots of the equation $F(0, y) = 0$, or, in geometrical language, is the number of points in which the curve $F = 0$ meets the axis of y at the origin; for shortness, let us call m the y-order of F at the origin. Then, adapting Weierstrass's general result to the case of two variables, we know that, corresponding to any positive number δ' less than a certain finite number, we can choose a finite positive number δ such that, for any value of x for which $|x| < \delta$, the equation $F(x, y) = 0$ has m roots y, and no more, for which $|y| < \delta'$. These roots are put *en évidence* by expressing F in the form

$$F(x,y) \equiv F_1(x,y)\, e^{G_1(x,y)},$$

where G_1 is an ordinary power series convergent in the region considered, and F_1 is a polynomial in y of order m, of the form

$$y^m + p_1 y^{m-1} + \dots + p_m,$$

where the cofficients are ordinary power series in x; and, when $|x| < \delta$, F_1 vanishes for m values of y, which satisfy the condition $|y| < \delta'$. The result holds also if F is a polynomial.

We now choose δ, δ', so as to satisfy the conditions of Weierstrass's theorem for the two functions F, ϕ of equation (1), and so that further, within the region defined by $|x| < \delta$, $|y| < \delta'$, f, ϕ only vanish simultaneously at the origin. This would, of course, be impossible if

* See also Harkness and Morley, *Treatise on the Theory of Functions*, § 88; and for a different method of treatment, Picard, *Traité d'Analyse*, Vol. II., pp. 241–246.

f, ϕ had a common factor. It is implied that F, Φ converge throughout the region thus defined. We can also, by a linear transformation, arrange so that x is not a factor of any of the four functions.

We now have
$$\left.\begin{aligned} F &\equiv F_1(x, y)\, e^{G_1(x, y)} \\ \phi &\equiv \phi_1(x, y)\, e^{G_2(x, y)} \end{aligned}\right\}, \tag{2}$$

where F_1, ϕ_1 are polynomials in y of orders m, n (these being the y-orders of F, ϕ at the origin), the coefficients being as before power series in x, and the exponentials are neither zero nor infinity, in the region considered.

Let us further resolve ϕ_1 into factors $(y-y_1)(y-y_2)\dots(y-y_n)$, where each root y_i is a function of x, which vanishes when $x=0$, and satisfies the condition $|y_i| < \delta'$, as long as $|x| < \delta$; any number of the functions y_i may be equal to one another.

The fundamental equation (1) now becomes
$$F_1(x, y)\, f(x, y) = \phi_1(x, y)\, \Phi\, e^{G(x, y)}, \tag{3}$$
where G is written for $G_2 - G_1$.

Now let x have any value in the region other than zero; then there are n values of y in the region for which the right-hand side of (3) vanishes; by hypothesis none of these pairs of values of x and y make f vanish; therefore there are at least n values of y in the region for which F_1 vanishes; therefore the order of F_1 is at least equal to n; therefore the y-order of F_1 is at least equal to n, that is $F_1(0, y)$ vanishes at least n times at the origin.

In equation (3), replace y by the function y_1 (which vanishes when $x=0$); then the right-hand side vanishes for all values of x in the region, and, since f can only vanish when $x=0$, F_1 vanishes when $0 < |x| < \delta$; but we have just proved that it vanishes when $x=0$. Therefore $F_1(x, y_1)$ vanishes for all values of x in the region.

Hence, for all such values of x,
$$\frac{F_1(x, y)}{y-y_1} = \frac{F_1(x, y) - F_1(x, y_1)}{y-y_1},$$

and, F_1 being a polynomial in y, this becomes, by actual division, a polynomial
$$y^{m-1} + p_1' y^{m-2} + \dots + p_{m-1}',$$

where the coefficients are rational integral functions of y_1 and of the coefficients of F_1. Repeating the process with each factor of ϕ_1 (which need not be distinct) we find that F_1/ϕ_1 is a polynomial in y

of order $m-n\ (\geqslant 0)$, the coefficients of which are rational integral functions of $y_1, y_2, \ldots, y_n$ and of the coefficient of F_1; but they are symmetric functions of $y_1, \ldots, y_n$; therefore they are rational integral functions of the coefficients of F_1, ϕ_1, which coefficients are ordinary power series in x; therefore so are also the coefficients in our new polynomial.

We have thus proved that F_1/ϕ_1 is an ordinary power series in x, y, convergent in our region; therefore also

$$F/\phi \equiv e^{-G_1(x,y)} F_1/\phi_1$$

is such a power series.

This proves the result required and supplies the gap in Halphen's argument.

§ 3.

The result thus obtained admits of some easy generalizations.

In the first place, if f, ϕ, instead of being polynomials, are ordinary power series in x, y convergent in the region considered, the proof is unaltered provided that the condition that f, ϕ should have no common algebraical factor is suitably modified. This condition was only used to ensure that f and ϕ should have no common zero except the origin in the region considered. If therefore f and ϕ are such that every point (α, β) for which they both vanish is separated from the origin by a finite interval, *i.e.*, satisfies an inequality $|\alpha| + |\beta| > \delta''$, where δ'' is some finite positive number, the proof holds without alteration.

Moreover, by treating any point (x_0, y_0) in the interior of the region of convergence of the four series f, ϕ, F, Φ as origin, provided that we can draw round x_0, y_0 a finite region, at no point of which (with the possible exception of x_0, y_0 itself) f and ϕ vanish simultaneously, we see that F/ϕ or Φ/f is expressible as an ordinary convergent power series $p(x-x_0, y-y_0)$. We may further treat the functions as analytical functions defined by the original power series and their "continuations," and the result will still hold for the interior of any region common to the four regions of continuity thus obtained. The general form of the theorem obtained may now be enunciated as follows:—

If f, ϕ, F, Φ be four analytical functions of two variables x, y, defined each by an ordinary power series and its continuations, and if throughout any region R common to the regions of continuity of the four functions, then: (1) the identity $Ff = \Phi\phi$ subsists, and (2) every

point common to $f=0$, $\phi=0$ is separated by a finite interval from every other such point, then the quotient $F/\phi \equiv \Phi/f$ is an analytical function, the region of continuity of which comprises all points lying in the interior (as distinguished from the boundary) of the region R, and which is therefore expansible in an ordinary power series $p\,(x-x_0,\, y-y_0)$, in the neighbourhood of every point x_0, y_0 in the interior of R.

[Since this paper was presented to the Society some references have been added and some small alterations have been made with the view to removing some obscurities and dealing with possible cases of exception. I am indebted to the referee for suggesting these improvements].

A Note on Minimal Surfaces. By T. J. I'A. Bromwich, St. John's College, Cambridge.

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We shall first investigate the condition for a minimal surface in tangential coordinates. That is, we attempt to find the condition that the envelope of the plane $lx+my+nz=p$ may be a minimal surface, where, of course, p is homogeneous of degree unity in l, m, n. The point of contact of the plane with its envelope is given by

$$x=\frac{\partial p}{\partial l},\quad y=\frac{\partial p}{\partial m},\quad z=\frac{\partial p}{\partial n},$$

and the corresponding normal is

$$\frac{\xi-x}{l}=\frac{\eta-y}{m}=\frac{\zeta-z}{n}=\frac{-\rho}{\sqrt{l^2+m^2+n^2}}=-\lambda, \text{ say},$$

where ξ, η, ζ are the current coordinates of a point on the normal at distance ρ from x, y, z. The value of ρ will give a principal radius of curvature of the envelope if ξ, η, ζ lies on the consecutive normal, or if

$$\left(\frac{\partial^2 p}{\partial l^2}-\lambda\right)dl+\frac{\partial^2 p}{\partial l\,\partial m}dm+\frac{\partial^2 p}{\partial l\,\partial n}dn-l\,d\lambda=0,$$

with two similar conditions.

For brevity, I write temporarily

$$a = \frac{\partial^2 p}{\partial l^2}, \text{ \&c., } \quad f = \frac{\partial^2 p}{\partial m \partial n}, \text{ \&c.,}$$

and we then have the above conditions in the form

$$\frac{(a-\lambda)\, dl + h\, dm + g\, dn}{l} = \frac{h\, dl + (b-\lambda)\, dm + f\, dn}{m}$$

$$= \frac{g\, dl + f\, dm + (c-\lambda)\, dn}{n}.$$

Also, since $\frac{\partial p}{\partial l}$ is homogeneous of order zero, we have, by Euler's theorem,

$$al + hm + gn = 0,$$

and so also

$$hl + bm + fn = 0,$$

$$gl + fm + cn = 0,$$

giving

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

Now multiplying up the equations for λ, we have

$$h\,(n\,dl) + (b-\lambda)(n\,dm) - (hl + bm)\, dn$$

$$= g\,(m\,dl) - (gl + cn)\, dm + (c-\lambda)\, m\, dn,$$

where fm, fn have been replaced by their values $-(hl+bm)$, $-(gl+cn)$.

This equation may be written

$$(a+b+c-\lambda)(m\,dn - n\,dm)$$

$$= a\,(m\,dn - n\,dm) + h\,(n\,dl - l\,dn) + g\,(l\,dm - m\,dl),$$

and similarly we have two more equations, but these are not independent, as they have been deduced from two equations originally.

We have, however, that

$$l\,(m\,dn - n\,dm) + m\,(n\,dl - l\,dn) + n\,(l\,dm - m\,dl) = 0,$$

and so, with

$$\kappa = \lambda - (a+b+c),$$

we have

$$\begin{vmatrix} a+\kappa & h & g \\ h & b+\kappa & f \\ l & m & n \end{vmatrix} = 0,$$

or $$l(G-\kappa g)+m(F-\kappa f)+n(C+\kappa\,\overline{a+b}+\kappa^2)=0,$$

where $F = gh-af$, &c. and $C = ab-h^2$, &c.

But, since
$$al+hm+gn=0,$$
$$hl+bm+fn=0,$$
we have
$$\frac{l}{G}=\frac{m}{F}=\frac{n}{C}.$$

Thus the equation above will reduce to
$$C\left[\kappa^2+\kappa(a+b+c)+A+B+C\right]-(\kappa+a+b)\,\Delta=0.$$

Hence, as $\Delta=0$, we have, in general,
$$\kappa^2+\kappa(a+b+c)+A+B+C=0,$$
or
$$\lambda^2-\lambda(a+b+c)+A+B+C=0.*$$

[*Note added* 15*th April*.—One of the referees remarks that this proof is not valid if $C=0$. I have examined this case, and it appears that then all the first minors of Δ vanish. It will also be found that the equation for κ reduces to
$$\kappa^2+\kappa(a+b+c)=0,$$
which is the limit of the ordinary equation when $A=0=B=C$.]

The quadratic just obtained appears to have been given by Mr. R. W. Genese (*Quart. Journ. of Math.*, Vol. XIII., 1875) in a form that is equivalent to this one. Mr. Genese's result is given without

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* Since $\Delta=0$, this may be written in the form
$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0,$$
where the zero root of λ is to be rejected.

Passing to the case of a plane curve, we have
$$\frac{\rho}{\sqrt{l^2+m^2}}=\frac{\partial^2 p}{\partial l^2}+\frac{\partial^2 p}{\partial m^2},$$
which is the same as the familiar form
$$\rho=q+\frac{d^2q}{d\phi^2},$$
on putting $$\cos\phi/l=\sin\phi/m=q/p.$$

proof, as an obvious extension of the problem in two dimensions; and in finding the envelope of the line $lx+my=p$ he assumes that

$$l^2+m^2=1.$$

It is clear, however, from the proof just given that it is not necessary to suppose that

$$l^2+m^2+n^2=1.$$

We have now the condition for a minimal surface

$$a+b+c=0,$$

or, written in full,

$$\frac{\partial^2 p}{\partial l^2}+\frac{\partial^2 p}{\partial m^2}+\frac{\partial^2 p}{\partial n^2}=0.$$

This leads at once to the familiar cases of the catenoid and helicoid; the simplest method of deduction seems to be by the introduction of new variables θ, ψ given by

$$\frac{\sin\theta\cos\psi}{l}=\frac{\sin\theta\sin\psi}{m}=\frac{\cos\theta}{n}=\frac{q}{p}.$$

So that q is the actual perpendicular from the origin on the enveloping plane, and θ, ψ give the direction of q in the usual manner of polar coordinates. We now have

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial q}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 q}{\partial\psi^2}+2q=0,$$

the ordinary equation satisfied by a spherical surface harmonic of order unity. This result can be verified immediately by finding the quadratic for the radii of curvature of the envelope of the plane

$$x\sin\theta\cos\psi+y\sin\theta\sin\psi+z\cos\theta=q,$$

where q is a function of θ, ψ. The simplest way of doing this appears to be by an application of the formulæ of moving axes in the manner due, I believe, to Mr. R. R. Webb; the process is analogous to that given by him for finding the strains of an elastic solid in polar coordinates (*Mess. of Math.*, February, 1882; Love's *Elasticity*, chap. vi.).

Passing to the solution of this equation, if q be the sum of a number of terms each of which is the product of a function of θ by a function of ψ, by the ordinary theory of Legendre's equation we shall have

$$q=(A\psi+B)\left[C\cos\theta+\cos\theta\tanh^{-1}(\cos\theta)-1\right]$$
$$+(D\cos\psi+E\sin\psi)\left[\sin\theta\tanh^{-1}(\cos\theta)+\cot\theta+F\sin\theta\right].$$

The particular case of a surface of revolution is seen to be

$$q = A' \cos\theta + B'\left[\cos\theta \tanh^{-1}(\cos\theta) - 1\right],$$

which is the catenoid.

Another simple case is

$$q = (A'\psi + B')\cos\theta,$$

which is the helicoid.

The surface

$$q = (D\cos\psi + E\sin\psi)\left[\sin\theta \tanh^{-1}(\cos\theta) + \cot\theta\right]$$

appears to be related to the catenoid, but I have not succeeded in recognizing it in any simple form.

Returning now to the form

$$\frac{\partial^2 p}{\partial l^2} + \frac{\partial^2 p}{\partial m^2} + \frac{\partial^2 p}{\partial n^2} = 0,$$

I find that Darboux (*Théorie générale des Surfaces*, Vol. I., Art. 195, p. 298) gives a solution of the minimal problem in the form

$$p = (l + im) f\left(\frac{l - im}{r - n}\right) - rf'\left(\frac{l - im}{r - n}\right),$$

where $$r^2 = l^2 + m^2 + n^2.$$

Darboux also includes terms in p which are found by changing the sign of i in the above. This result he deduces from the expressions for the point-coordinates of points on a minimal surface.

On examining his solution it appeared to be deducible from the solutions of $\nabla^2 v = 0$ given by Prof. Forsyth (*Mess. of Math.*, Vol. XXVII.), but not of the most general type that could be so found. I have accordingly attempted to deduce solutions of $\nabla^2 v = 0$ of this type; my results are given in the paper following.

I find that, if P, Q, R be functions of a variable u such that

$$P^2 + Q^2 + R^2 = 0,$$

$$P'^2 + Q'^2 + R'^2 = 1,$$

and u be given by $$lP + mQ + nR = 0,$$

then a corresponding value of p is

$$(lP'' + mQ'' + nR'') f(u) - (lP' + mQ' + nR') f'(u),\text{*}$$

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* See p. 290 below.

which agrees with Darboux's solution on putting

$$P = (u^2-1)/2,$$
$$Q = i\,(u^2+1)/2,$$
$$R = u,$$

for then we have $(l+im)\,u^2+2nu+(-l+im) = 0$

or $$u = \frac{\pm r-n}{l+im} = \frac{l-im}{\pm r+n},$$

also $$lP''+mQ''+nR'' = l+im,$$
$$lP'+mQ'+nR' = \pm r,$$

and on substitution these will give the solution quoted.

On some Solutions of $\nabla^2 v = 0$. *By* T. J. I'A. BROMWICH, St. John's College, Cambridge. Received February 6th, 1899. Read February 9th, 1899.

Prof. Forsyth in his paper on this subject (*Mess. of Math.*, Vol. XXVII.) shows how to solve $\nabla^2 v = 0$ and allied equations when the solution of a certain functional equation is known. I recall briefly his results, stating them for three variables only, though, as he shows, the solutions hold for any number of variables.

He takes p, q, r three functions of a variable u such that

$$p^2+q^2+r^2 = 0,$$

and u is given by $$au = px+qy+rz,$$

where a is an arbitrary constant.

He then shows that we may take, as a solution of $\nabla^2 v = 0$,

$$v = f(u) + \frac{1}{\Delta}\,g(u),$$

where f, g are perfectly arbitrary functions and

$$\Delta = a-p'x-q'y-r'z.$$

Here, as throughout my work, accents indicate differential coefficients with respect to u.

It became clear to me in the course of the investigation contained in the preceding paper that solutions ought to exist of the type

$$v = Xf(u) + Yf'(u),$$

where X, Y are not necessarily functions of u alone. I now show how to effect a determination of the forms of X, Y.

On differentiation we have

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 X}{\partial x^2} f + \left[2\frac{\partial u}{\partial x}\frac{\partial X}{\partial x} + X\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 Y}{\partial x^2}\right] f'$$
$$+ \left[X\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial u}{\partial x}\frac{\partial Y}{\partial x} + Y\frac{\partial^2 u}{\partial x^2}\right] f'' + Y\left(\frac{\partial u}{\partial x}\right)^2 f'''.$$

Now, on addition of this and two similar terms, we get

$$\nabla^2 v = (\nabla^2 X) f + \left[2\left(\frac{\partial u}{\partial x}\frac{\partial X}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial X}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial X}{\partial z}\right) + \nabla^2 Y\right] f'$$
$$+ 2\left[\frac{\partial u}{\partial x}\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial Y}{\partial z}\right] f'',$$

for, as shown by Prof. Forsyth, we have

$$\nabla^2 u = 0$$

and
$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 0.$$

Hence, to get a solution of the type indicated, we may take

$$\nabla^2 X = 0,$$
$$\nabla^2 Y + 2\left(\frac{\partial u}{\partial x}\frac{\partial X}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial X}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial X}{\partial z}\right) = 0,$$
$$\frac{\partial u}{\partial x}\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial Y}{\partial z} = 0.$$

In the application originally contemplated, it seemed that Y would be a function of Δ only, and I restrict myself to this case.

Taking then $$Y = F(\Delta),$$

we have that the third equation of condition is at once satisfied, for

$$\frac{\partial u}{\partial x}\frac{\partial \Delta}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial \Delta}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial \Delta}{\partial z} = 0.$$

Next $\nabla^2 Y = \left[\left(\frac{\partial\Delta}{\partial x}\right)^2 + \left(\frac{\partial\Delta}{\partial y}\right)^2 + \left(\frac{\partial\Delta}{\partial z}\right)^2\right] F''(\Delta) + (\nabla^2\Delta)\, F'(\Delta),$

and we have $$-\frac{\partial\Delta}{\partial x} = p' + \frac{p}{\Delta}(xp'' + yq'' + zr''),$$

on using the result $$\frac{\partial u}{\partial x} = \frac{p}{\Delta}.$$

Hence we have

$$\left(\frac{\partial\Delta}{\partial x}\right)^2 + \left(\frac{\partial\Delta}{\partial y}\right)^2 + \left(\frac{\partial\Delta}{\partial z}\right)^2 = p'^2 + q'^2 + r'^2,$$

and we find, on differentiating again and adding,

$$\begin{aligned} \nabla^2\Delta &= -2\,(pp'' + qq'' + rr'')/\Delta \\ &= \quad 2\,(p'^2 + q'^2 + r'^2)/\Delta, \end{aligned}$$

for $$pp' + qq' + rr' = 0,$$

and, differentiating again, we find

$$(p'^2 + q'^2 + r'^2) + (pp'' + qq'' + rr'') = 0.$$

Thus $$\nabla^2 Y = (p'^2 + q'^2 + r'^2)\left[F''(\Delta) + \frac{2}{\Delta}F'(\Delta)\right].$$

Next, for a type to be taken as X, I use

$$X = xP + yQ + zR,$$

where P, Q, R are also functions of u only. We then have, on differentiation,

$$\frac{\partial X}{\partial x} = P + (xP' + yQ' + zR')\frac{\partial u}{\partial x},$$

$$\begin{aligned} \frac{\partial^2 X}{\partial x^2} = 2P'\frac{\partial u}{\partial x} &+ (xP'' + yQ'' + zR'')\left(\frac{\partial u}{\partial x}\right)^2 \\ &+ (xP' + yQ' + zR')\frac{\partial^2 u}{\partial x^2}. \end{aligned}$$

Thus $$\nabla^2 X = 2\left(P'\frac{\partial u}{\partial x} + Q'\frac{\partial u}{\partial y} + R'\frac{\partial u}{\partial z}\right),$$

the other terms vanishing as before, and this expression

$$= 2\,(P'p + Q'q + R'r)/\Delta.$$

Also $$\frac{\partial X}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial X}{\partial y}\frac{\partial u}{\partial y}+\frac{\partial X}{\partial z}\frac{\partial u}{\partial z}=P\frac{\partial u}{\partial x}+Q\frac{\partial u}{\partial y}+R\frac{\partial u}{\partial}$$
$$=(Pp+Qq+Rr)/\Delta;$$

whence we have that the form assumed for v will give a solution provided
$$P'p+Q'q+R'r=0$$

and $$2(Pp+Qq+Rr)+(p'^2+q'^2+r'^2)\left[2F'(\Delta)+\Delta F''(\Delta)\right]=0.$$

Now Δ cannot be expressed as a function of u only, but $(Pp+Qq+Rr)$ and $(p'^2+q'^2+r'^2)$ both involve no other variable than u; consequently we must have
$$\Delta F''(\Delta)+2F'(\Delta)=2A,$$

where A is a constant.

Hence $$F(\Delta)=A\Delta+B+C/\Delta,$$

by integration, B, C being constants of integration. Thus we now have only to satisfy
$$P'p+Q'q+R'r=0$$

and $$Pp+Qq+Rr=-A(p'^2+q'^2+r'^2).$$

Differentiating the second of these and using the first, we find
$$Pp'+Qq'+Rr'=-2A(p'p''+q'q''+r'r'').$$

Now, since $$p^2+q^2+r^2=0,$$
$$pp'+qq'+rr'=0,$$

we have $$\frac{qr'-q'r}{p}=\frac{rp'-r'p}{q}=\frac{pq'-p'q}{r}=\kappa,\text{ say},$$

and then $$p'^2+q'^2+r'^2=\left[(r'p+\kappa q)^2+(r'q-\kappa p)^2+r^2r'^2\right]/r^2$$
$$=-\kappa^2.$$

Hence th equations for P, Q, R are
$$Pp+Qq+Rr=A\kappa^2,$$
$$Pp'+Qq'+Rr'=2A\kappa\kappa,$$

and $$Px+Qy+Rz=X.$$

These give, on eliminating P, Q,

$$\begin{vmatrix} p & q & A\kappa^2 - Rr \\ p' & q' & 2A\kappa\kappa' - Rr' \\ x & y & X - Rz \end{vmatrix} = 0$$

or
$$(X-Rz)(pq'-p'q)+(A\kappa^2-Rr)(p'y-q'x)$$
$$+(2A\kappa\kappa'-Rr')(qx-py)=0.$$

Here the coefficient of R is

$$-\begin{vmatrix} p & q & r \\ p' & q' & r' \\ x & y & z \end{vmatrix},$$

which is
$$-\kappa\,(px+qy+rz) = -\kappa au.$$

Hence we have

$$\kappa rX - \kappa auR + A\kappa^2\,(p'y-q'x) + 2A\kappa\kappa'\,(qx-py) = 0,$$

or
$$rX = auR + A\kappa\,(q'x-p'y) + 2A\kappa'\,(py-qx).$$

This solution for X is not symmetrical; a symmetrical form can be deduced by the addition of three such results, but their sum does not appear to simplify to any extent.

Hence a solution of $\nabla^2 v = 0$

is
$$v = \frac{1}{r}\left[auR + A\kappa\,(q'x-p'y) + 2A\kappa'\,(py-qx)\right] f(u)$$
$$+ (A\Delta + B + C/\Delta)\, f'(u),$$

and it is clear that the terms in B, C are included in Prof. Forsyth's types of solution; further, since $uRf(u)/r$ is an arbitrary function of u, this term may be considered as also included in his solutions.

Thus the investigation indicates a solution of the new type

$$v = \left[\frac{\kappa}{r}\,(q'x-p'y) + 2\,\frac{\kappa'}{r}\,(py-qx)\right] f(u) + \Delta f'(u).$$

Turning to the case worked out by Prof. Forsyth, we may take

$$p = (u^2-1)/2,$$
$$q = i\,(u^2+1)/2,$$
$$r = u,$$

and then
$$u = (a-z+\rho)/(x+iy),$$
where
$$\rho = \pm\sqrt{(a-z)^2+x^2+y^2}.$$
We find
$$\Delta = -\rho \quad \text{and} \quad \kappa = -i,$$
so that
$$v = (x+iy)f\left(\frac{a-z+\rho}{x+iy}\right) - \rho f'\left(\frac{a-z+\rho}{x+iy}\right). \tag{A}$$

It will be seen that, if we differentiate with respect to x any of the results found, we shall get solutions of

$$\nabla^2 v = 0.*$$

Consider then $\dfrac{\partial}{\partial x} f(u) = f'(u)\dfrac{\partial u}{\partial x} = \dfrac{1}{\Delta} pf'(u).$

But, since p is a function of u, this simply leads to Prof. Forsyth's second type of solution. Differentiating again, we have

$$\frac{\partial}{\partial x}\frac{f(u)}{\Delta} = \frac{f'(u)}{\Delta}\frac{p}{\Delta} + \frac{f(u)}{\Delta^3}\left[p' + \frac{p}{\Delta}(xp''+yq''+zr'')\right]$$
$$= \frac{1}{\Delta^3}\left[g'(u)\,\Delta + (xp''+yq''+zr'')\,g(u)\right],$$
where
$$g(u) = pf(u).$$

Now, on substituting the particular set of values for p, q, r given above, we find that this solution is of the form

$$\frac{1}{\rho^3}\left[(x+iy)\,g(u) - \rho g'(u)\right].$$

Consequently the type (A) is of the form

$$\rho^3\frac{\partial^2}{\partial x^2}f(u)$$

in the particular case just referred to. Of course this type of solution could be anticipated by the general result that, if v be a solution of zero order, then $\rho^{2n+1}\dfrac{\partial^{n+1}v}{\partial x^{n+1}}$ is a solution of order n.

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* *Note added* 15*th April*.—One of the referees remarks that Prof. Burnside has alluded to the series of solutions so found in a paper "On Equipotentials" in the volume of the *Messenger* which contains Prof. Forsyth's paper.

On trying to generalize this type of solution, it seemed probable that the form to use would be

$$\Delta^3 \frac{\partial^2}{\partial x^2} f(u),$$

but it will be found that this will only satisfy the conditions found above when

$$p'p''+q'q''+r'r'' = 0,$$

or when

$$p'^2+q'^2+r'^2 = \text{const.}$$

It will always be possible to satisfy this condition, for, if the values of p, q, r as originally found do not lead to the result, consider

$$\left[\frac{d}{du}(p\theta)\right]^2 + \left[\frac{d}{du}(q\theta)\right]^2 + \left[\frac{d}{du}(r\theta)\right]^2,$$

where θ is any function of u; this will be seen to be

$$\theta^2 (p'^2+q'^2+r'^2),$$

and hence, if we take

$$\theta = i/\kappa,$$

we shall find that

$$p_1 = ip/\kappa, \quad q_1 = iq/\kappa, \quad r_1 = ir/\kappa$$

will satisfy

$$p_1'^2+q_1'^2+r_1'^2 = 1.$$

I shall now attempt to find in what cases $\Delta^{2n+1} \dfrac{\partial^{n+1}}{\partial x^{n+1}} f(u)$ is a solution of Laplace's equation. To attack this, it will be simplest to find the condition that

$$\nabla^2 (\Delta^m Z) = 0,$$

when

$$\nabla^2 Z = 0.$$

On performing the differentiation, we find that

$$(m+1)\, Z + 2\Delta \left(\frac{\partial \Delta}{\partial x}\frac{\partial Z}{\partial x} + \frac{\partial \Delta}{\partial y}\frac{\partial Z}{\partial y} + \frac{\partial \Delta}{\partial z}\frac{\partial Z}{\partial z}\right) = 0.$$

Now, when Z is $\dfrac{\partial^{n+1}}{\partial x^{n+1}} f(u)$, it will contain x explicitly and also implicitly in u, Δ; in fact,

$$Z = \phi\,(x, y, z, u, \Delta),$$

and

$$\frac{\partial Z}{\partial x} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial \Delta}\frac{\partial \Delta}{\partial x},$$

where $\dfrac{\partial\phi}{\partial x}$ is to be understood as derived only from the terms explicitly containing x.

On substituting this and the similar expressions in the condition obtained previously, we get

$$\frac{m+1}{2\Delta}\varphi+\frac{\partial\phi}{\partial\Delta}+\left(\frac{\partial\Delta}{\partial x}\frac{\partial\phi}{\partial x}+\frac{\partial\Delta}{\partial y}\frac{\partial\phi}{\partial y}+\frac{\partial\Delta}{\partial z}\frac{\partial\phi}{\partial z}\right)=0,$$

for
$$\frac{\partial u}{\partial x}\frac{\partial\Delta}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial\Delta}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial\Delta}{\partial z}=0,$$

and
$$\left(\frac{\partial\Delta}{\partial x}\right)^2+\left(\frac{\partial\Delta}{\partial y}\right)^2+\left(\frac{\partial\Delta}{\partial z}\right)^2=p'^2+q'^2+r'^2=1,$$

it being supposed that p, q, r are determined as just explained.

Now we have seen that

$$\frac{\partial^2}{\partial x^2}f(u)=\frac{g'(u)}{\Delta^2}+\frac{g(u)}{\Delta^3}(p''x+q''y+r''z),$$

so consider the next differential coefficient,

$$\phi=\frac{\partial^3}{\partial x^3}f(u)=\frac{h''(u)}{\Delta^3}+\frac{3h'(u)}{\Delta^4}(p''x+q''y+r''z)$$
$$+\frac{h(u)}{\Delta^5}\left[3(p''x+q''y+r''z)^2+(p'''x+q'''y+r'''z)\Delta\right],$$

where we put
$$h(u)=pg(u),$$

and we now find that

$$-\frac{\partial\phi}{\partial}-\frac{\partial\Delta}{\partial x}\frac{\partial\phi}{\partial x}-\frac{\partial\Delta}{\partial y}\frac{\partial\phi}{\partial y}-\frac{\partial\Delta}{\partial z}\frac{\partial\phi}{\partial z}$$
$$=\frac{3h''(u)}{\Delta^4}+\frac{9h'(u)}{\Delta^5}(p''x+q''y+r''z)+\frac{9h(u)}{\Delta^6}(p''x+q''y+r''z)^2$$
$$+\frac{h(u)}{\Delta^5}\left[4(p'''x+q'''y+r'''z)+\Delta(p'p'''+q'q'''+r'r''')\right],$$

where we have used the fact that here

$$pp'''+qq'''+rr'''=0,$$

for, since
$$p'^2+q'^2+r'^2=1=-(pp''+qq''+rr''),$$

we have
$$p'p''+q'q''+r'r''=0$$

and $$(p'p''+pp''')+(q'q''+qq''')+(r'r''+rr''')=0,$$

giving $$pp'''+qq'''+rr'''=0.$$

Returning to the condition that $\phi\,\Delta^m$ may be a solution of $\nabla^2 v = 0$, it will be seen that $m+1 = 6$ will satisfy the condition, provided

$$p'''x+q'''y+r'''z+\Delta\,(p'p'''+q'q'''+r'r''')=0.$$

Hence $$\frac{p'''}{p'}=\frac{q'''}{q'}=\frac{r'''}{r'}=p'p'''+q'q'''+r'r'''$$

and $$a\,(p'p'''+q'q'''+r'r''')=0,$$

Thus either $p'''=0$, $q'''=0$, $r'''=0$; or we may take $a=0$, and then $\frac{p'''}{p'}=\frac{q'''}{q'}=\frac{r'''}{r'}$ will satisfy all the conditions.

In the first case p, q, r are each quadratic functions of u, and here the most general values can be only linear functions of the particular set

$$p=(u^2-1)/2, \quad q=i\,(u^2+1)/2, \quad r=u\,;$$

and the solution of $\nabla^2 v=0$ so found can be deduced from that particular case by a transformation of coordinates.

In the second case, when $a=0$, the functions involved are all homogeneous in x, y, z, and we can apply the ordinary result of spherical harmonics, that, if Z be a solution of $\nabla^2 v = 0$, homogeneous in x, y, z of degree $-(n+1)$, then the corresponding solution of degree n is $(x^2+y^2+z^2)^{n+\frac{1}{2}}\,Z$.

We are thus led to the conclusion that the form $\Delta^{2n+1}\dfrac{\partial^{n+1}}{\partial x^{n+1}}f(u)$ for values of $n>1$ is a solution of $\nabla^2 v=0$ only when p, q, r are quadratic functions of u; and that, in the particular case of $a=0$, we may use the type $(x^2+y^2+z^2)^{n+\frac{1}{2}}\dfrac{\partial^{n+1}}{\partial x^{n+1}}f(u)$. It will be seen, moreover, that the first case is virtually included in the second, for we can make a zero by shifting the origin, when p, q, r are only quadratic functions of u, and then

$$\Delta^2 = x^2+y^2+z^2.$$

Thus the only new form we can find in this way would seem to be the one first considered, namely,

$$v=\left[\frac{\kappa}{r}\,(q'x-p'y)+\frac{2\kappa'}{r}\,(py-qx)\right]f(u)+\Delta f'(u),$$

which is the same as

$$v = (p''x+q''y+r''z)\,f(u)+\Delta f'(u),$$

when we make $$p'^2+q'^2+r'^2 = 1.$$

The other forms investigated will be the same as those given by Sommerfeld in a paper "On Diffraction," in the *Mathematische Annalen* (t. XLVII., 1896), in the form of contour integrals, where they are deduced by a stereographic projection.

The Irreducible Concomitants of any Number of Binary Quartics.

By A. YOUNG. Received and read February 9th, 1899.

The irreducible system is here arrived at by first finding the irreducible system of types and then the number of independent forms belonging to each type for a system of N quartics. Two concomitants are said to be of the same type when they can be obtained from the same form by polarization. For the purpose of discussing the system of types, a type is regarded as being of the first degree in the coefficients of each of the quantics concerned. The finiteness of the irreducible system of types has been established by Prof. Peano.* He proves that the complete system of concomitants for any number of binary n-ics may be obtained from the system for n n-ics by polarization alone; with the one possible exception of invariants of the type

$$\begin{vmatrix} A_0 & A_1 & \dots & A_n \\ B_0 & B_1 & \dots & B_n \\ \dots & & \dots & \dots \\ K_0 & K_1 & \dots & K_n \end{vmatrix}.$$

In other words, every type of a binary n-ic which furnishes no irreducible form for n n-ics is reducible, with the possible exception just mentioned. It was with the help of this proposition that some of the reductions for the quartic were first arrived at; however, other

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* *Atti di Torino*, t. XVII., p. 580.

methods have proved shorter. The latter part of his paper is devoted to the discovery of the cubic types. From the fact that

$$\begin{vmatrix} A_0 & A_1 & A_2 & A_3 \\ B_0 & B_1 & B_2 & B_3 \\ C_0 & C_1 & C_2 & C_3 \\ D_0 & D_1 & D_2 & D_3 \end{vmatrix}$$

is reducible, it is shown that all the types occur in the system for two cubics. His results are—

The irreducible system for N cubics belongs to 10 types, as follows:—

	Type.	Simplest Form.	Number of Forms.
I.	${}_3C_1$	One of the cubics	N
II.	${}_4C_2$	Jacobian of two cubics	$\binom{N}{2}$
III.	${}_2C_2$	Hessian of one cubic	$\binom{N+1}{2}$
IV.	I_2	Third transvectant of two cubics	$\binom{N}{2}$
V.	${}_3C_3$	Covariant order 3 of one cubic	$\binom{N+2}{3}$
VI.	${}_1C_3$	Second transvectant of I. and III.	$2\binom{N+1}{3}$
VII.	I_4	Discriminant of one cubic	$\binom{N+3}{4}$
VIII.	${}_2C_4$	Jacobian of two forms III.	$3\binom{N+2}{4}$
IX.	${}_1C_5$	First transvectant of III. and VI.	$4\binom{N+3}{5}$
X.	I_6	Resultant of two forms VI.	$\binom{N+4}{6}$

For the quartic, I have first expressed the types in symbols based on the quadratic. To do this, it is proved that the types of a binary mn-ic can be expressed in symbols based on the n-ic; the symbolical factors being of the form of n-ic types; just as, in ordinary symbolical

algebra, the concomitants of the m-ic are expressed in symbols based on the linear form. It is easy then to show that there is only one type to be considered, of given degree and order. Writing this $(abc \dots k)$, the fundamental identities give relations of the form

$$(1+S_1+S_2+\dots+S_k)(abc \dots k) = R,$$

where R stands for reducible terms, and $S_1, S_2, \dots, S_k$ are certain substitutions.

The chief advantage obtained from quadratic symbols lies in the possibility of using symbolical operators, with the help of which relations between forms of one degree and order may be obtained from relations between forms of the same order but of one degree lower.

The invariant type of highest degree I_6 has been expressed in terms of determinants of five rows and columns; by means of this a number of syzygies may be at once written down, in fact $I_6 P$ equals a sum of products of forms, there being at least three forms in each product, where P is an irreducible form of any type except I_2 and I_3.

1. Consider any simultaneous system of binary nm-ics,

$$(A_0, A_1, \dots, A_{mn} \between x_1, x_2)^{mn} \equiv a_{x^m}^n,$$

$$(B_0, B_1, \dots, B_{mn} \between x_1, x_2)^{mn} \equiv b_{x^m}^n,$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

where $$a_{x^m} \equiv (a_0, a_1, \dots, a_m \between x_1, x_2)^m,$$

and the identities are taken to define the relations between the symbolical letters $a_0, a_1, \dots,$ and the actual coefficients. Let $f(A, B, \dots, K)$ be a type belonging to this system; writing in this for $A_0, \dots$ their values in terms of the symbolical letters, $f(A, B, \dots, K)$ takes the form $\phi(a, b, \dots, k)$, say. Now make any linear transformation, and denote by dashed letters the coefficients of the transformed quantics; then

$$f(A', B', \dots, K') = \mu f(A, B, \dots, K),$$

where μ is a power of the determinant of transformation; hence also

$$\phi(a', b', \dots, k') = \mu \phi(a, b, \dots, k).$$

Therefore ϕ is a concomitant of the m-ics $a_{x^m}, b_{x^m}, \dots, k_{x^m}$; and hence ϕ can be expressed as a sum of products each factor of which is of the same form as an irreducible type of the m-ic.

It is necessary now to show how to proceed from a symbolical product P to the form F in the actual coefficients, which it represents. Let

$$a_{x^m} \equiv \alpha_x^m, \quad b_{x^m} \equiv \beta_x^m, \quad \ldots;$$

then the same result will be obtained by writing in F the coefficients of the first quantic in terms of $a_0, a_1, \ldots, a_m$, and then putting

$$a_r = \alpha_1^{m-r} \alpha^r$$

as by writing

$$A_r = \alpha_1^{mn-r} \alpha_2^r$$

directly in that type. Hence the result of writing in P

$$a_r = \alpha_1^{m-r} \alpha_2^r, \ \ldots, \ b_r = \beta_1^{m-r} \beta_2^r, \ \ldots$$

is the same mn-ic type expressed in linear symbols; the step from these to the actual coefficients presents no difficulty. As a matter of fact, an mn-ic type, when expressed in m-ic symbols, will rarely consist of a single symbolical product; still, given any symbolical product P—of the right degree in the symbols—a type F of the mn-ic may be arrived at, in general, with the help of linear symbols, which is such that when expressed in m-ic symbols it becomes $P+Q$, where the effect of substituting linear for m-ic symbols is to make Q vanish.

Hence products which vanish when the change is made to linear symbols may be ignored, and every other symbolical product of the right degree in the symbols may be regarded as giving a type of the mn-ic.

Let $(a, b, c, \ldots)$ be an m-ic type; then among the factors which may occur are forms like $(a, a, c, \ldots)$; a factor of this kind may be reducible, or when linear symbols are introduced it may vanish; in either case there is no need to consider it.

The lineo-linear type (a, b) is a case in point; a product with a factor (a, a) may always be ignored.

2. The types of a quadratic are—

$$a_0 b_2 + a_2 b_0 - 2a_1 b_1 \equiv [ab], \quad a_0 x_1^2 + 2a_1 x_1 x_2 + a_2 x_2^2 \equiv a_{x^2},$$

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{vmatrix} \equiv |\, abc \,|, \quad \begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ x_2^2 & -x_1 x_2 & x_1^2 \end{vmatrix} \equiv |\, abx^2 \,|;$$

no symbolical factor in quadratic symbols need be considered in which the same letter occurs twice.

The fundamental identities are

$$2\,|\,abc\,|\;|\,def\,| = \begin{vmatrix} [ad] & [ae] & [af] \\ [bd] & [be] & [bf] \\ [cd] & [ce] & [cf] \end{vmatrix}, \qquad \text{I.}$$

$$[ae]\,|\,bcd\,| + [be]\,|\,cad\,| + [ce]\,|\,abd\,| = [de]\,|\,abc\,|, \qquad \text{II.}$$

$$\begin{vmatrix} [ab] & [ad] & [af] & [ah] \\ [cb] & [cd] & [cf] & [ch] \\ [eb] & [ed] & [ef] & [eh] \\ [gb] & [gd] & [gf] & [gh] \end{vmatrix} = \begin{vmatrix} a_0 & a_1 & a_2 & 0 \\ c_0 & c_1 & c_2 & 0 \\ e_0 & e_1 & e_2 & 0 \\ g_0 & g_1 & g_2 & 0 \end{vmatrix} \begin{vmatrix} b_2 & -2b_1 & b_0 & 0 \\ d_2 & -2d_1 & d_0 & 0 \\ f_2 & -2f_1 & f_0 & 0 \\ h_2 & -2h_1 & h_0 & 0 \end{vmatrix} = 0. \qquad \text{III.}$$

The identities for the forms a_{x^2}, $|\,abx^2\,|$ may be obtained at once from these, since $[ad]$, $|\,abd\,|$ become respectively a_{x^2}, $|\,abx^2\,|$ if x_2^2, $-x_1x_2$, x_1^2 are written for d_0, d_1, d_2.

From I. it follows that no product need be discussed in which more than one factor of either of the forms $|\,abc\,|$, $|\,abx^2\,|$ occurs. In quartic types each symbolical letter must occur twice in a product; hence, if there is a factor $|\,abc\,|$, there is also a factor e_{x^2}; but, from II.,

$$|\,abc\,|\,e_{x^2} = [ae]\,|\,bcx^2\,| + [be]\,|\,cax^2\,| + [ce]\,|\,abx^2\,|;$$

hence, for the quartic, factors $|\,abc\,|$ need not be considered. The invariant forms then are

$$[ab]^2,$$
$$[ab][bc][ca],$$
$$[ab][bc][cd][da],$$
$$\dots \quad \dots \quad \dots \quad \dots .$$

We shall find it convenient to use the notation

$$(abca) \equiv [ab][bc][ca],$$
$$(abcda) \equiv [ab][bc][cd][da],$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots;$$

then $$(abcd \dots ka) \equiv (bcd \dots kab) \equiv (ak \dots dcba).$$

The general form of covariant type of order 2 is

$$[ab][bc]\ldots[hk]\ \mid kax^2 \mid ,$$

which will be denoted by $(ab\ldots hk)$, and here

$$(ab\ldots hk) = -(kh\ldots ba).$$

Covariants order 4 are of the form

$$[ab][bc]\ldots[hk]\ h_{x^2}k_{x^2} \equiv (abc\ldots hkx^2a)\ ;$$

and so on.

3. *The Invariants.*—Multiply III. by $[bc][de][fg][ha]$, and expand; then

$$(abcdefgha)+(afgbcdeha)+(adefgbcha)+(abcfgdeha)+(afgdebcha)$$
$$+(adebcfgha) = R,$$

where R is used to express reducible terms, and terms which have factors of the form $[aa]$. The above may be conveniently written

$$[a,\ bc,\ de,\ fg,\ ha] = R.$$

From the identity

$$\begin{vmatrix} [ab] & [ac] & [ad] & [aa] \\ [bb] & [bc] & [bd] & [ba] \\ [cb] & [cc] & [cd] & [ca] \\ [db] & [dc] & [dd] & [da] \end{vmatrix} = 0,$$

we deduce

$$[a\ldots a] \equiv (abcda)+(adbca)+(acdba)+(abdca)+(adcba)+(acbda)$$
$$= R.$$

The other relations obtainable may be written

$$[a\ldots ea] = R,\quad [a,\ bc,\ d,\ e,\ a] = R,\quad [a,\ bc,\ de,\ f,\ a] = R,\ \&c.$$

The types I_2 and I_3, viz., $[ab]^2$ and $(abca)$, are unaffected by the fundamental identity; hence for N quartics there are $\binom{N+1}{2}$ and $\binom{N+2}{3}$ independent irreducible invariants respectively of these types.

For I_4 there is one equation,

$$[a\ldots a] = R,$$

or

$$2(abcda)+2(adbca)+2(acdba) = R.$$

There are then only two independent forms having the same letters. Let $(A^pB^q \ldots)\, I_k$ denote an invariant of the type I_k which is of degree p in the coefficients of the quartic A, of degree q in the coefficients of the quartic B, and so on. Then there are two independent irreducible forms $(ABCD)\, I_4$, one form $(ABC^2)\, I_4$, and one form $(A^2B^2)\, I_4$; in other cases the invariant is reducible. Hence for N quartics there are $\binom{N}{2}+3\binom{N}{3}+2\binom{N}{4} = \binom{N+2}{4}+\binom{N+1}{4}$ invariants of the type I_4.

There are twelve possible forms $(ABCDE)\, I_5$, and ten relations between them, all of which are of the form

$$[ab \ldots a] = R.$$

Simpler relations are obtained thus:—

$$[ab \ldots a]+[ae \ldots a]-[ac \ldots a]-[ad \ldots a]$$
$$\equiv 2\,(abcdea)+2\,(abdcea)-2\,(acbeda)-2\,(acebda) = R. \qquad \text{IV.}$$

In connexion with this type consider the form

$$|\, ABCDE \,| \equiv \begin{vmatrix} A_0 & A_1 & A_2 & A_3 & A_4 \\ B_0 & B_1 & B_2 & B_3 & B_4 \\ C_0 & C_1 & C_2 & C_3 & C_4 \\ D_0 & D_1 & D_2 & D_3 & D_4 \\ E_0 & E_1 & E_2 & E_3 & E_4 \end{vmatrix}$$
$$= (\alpha\beta)(\alpha\gamma)(\beta\gamma)(\alpha\delta)(\beta\delta)(\gamma\delta)(\alpha\epsilon)(\beta\epsilon)(\gamma\epsilon)(\delta\epsilon).$$

where $\qquad a_x^2 \equiv (A_0,\, A_1,\, A_2,\, A_3,\, A_4 \between x_1,\, x_2)^4 \equiv \alpha_x^4$, &c.

Using the identity

$$3\,(\alpha\beta)(\alpha\gamma)(\beta\gamma)(\alpha\delta)(\beta\delta)(\gamma\delta) = -(\alpha\beta)^3(\gamma\delta)^3-(\alpha\gamma)^3(\delta\beta)^3-(\alpha\delta)^3(\beta\gamma)^3,$$

we obtain $\qquad 6\,|\, ABCDE \,|$

$$= -2\,(\alpha\beta)^3(\gamma\delta)^3(\alpha\epsilon)(\beta\epsilon)(\gamma\epsilon)(\delta\epsilon)-\ldots$$
$$= -(\alpha\beta)^2(\gamma\delta)^2 \begin{vmatrix} (\alpha\gamma)^2 & (\alpha\delta)^2 & (\alpha\epsilon)^2 \\ (\beta\gamma)^2 & (\beta\delta)^2 & (\beta\epsilon)^2 \\ (\epsilon\gamma)^2 & (\epsilon\delta)^2 & 0 \end{vmatrix} -\ldots$$
$$= (abedca)+(abdcea)+(acebda)+(acbdea)+(adecba)+(adcbea)$$
$$-(abecda)-(abcdea)-(acedba)-(acdbea)-(adebca)$$
$$-(adbcea)+R,$$

since $(abcdea)$ and $(\alpha\beta)^2(\beta\gamma^2)(\gamma\delta^2)(\delta\epsilon^2)(\epsilon\alpha)^2$ represent the same type.

Hence, from equation IV.,

$$\begin{aligned}
&(abdcea)-(acbeda)+R = (acebda)-(abcdea)+R \\
&= (acbdea)-(adceba)+R = (adecba)-(acdbea)+R \\
&= (adcbea)-(abdeca)+R = (abedca)-(adbcea)+R \\
&= |\,ABCDE\,| \\
&= \tfrac{1}{3}\left\{(abdcea)+(acebda)+(acbdea)+(adecba)+(adcbea)\right. \\
&\qquad\qquad\qquad\qquad \left.+(abedca)\right\}+R, \qquad \text{V.}
\end{aligned}$$

this last in virtue of

$$[ab\,\ldots\,a]+[ac\,\ldots\,a]+[ad\,\ldots\,a]+[ae\,\ldots\,a] = R.$$

These relations include all those from which we started; further they are independent; hence, there are six independent irreducible invariants $(ABCDE)$ I_5. For N quartics the number of independent irreducible forms I_5 is

$$6\binom{N}{5}+8\binom{N}{4}+3\binom{N}{3} = 3\binom{N+2}{5}+2\binom{N+1}{5}+\binom{N}{5}.$$

For I_6 there are two systems of equations, viz.,

$$[abc\,\ldots\,a] = R \quad \text{and} \quad [a,\, bc,\, d,\, e,\, fa] = R.$$

A third equation, which, though not independent of these, is useful, may be found thus:

$$2\,|\,abc\,|\;|\,def\,| = \begin{vmatrix} [ad] & [ae] & [af] \\ [bd] & [be] & [bf] \\ [cd] & [ce] & [cf] \end{vmatrix};$$

expand the determinant and square both sides

$$\begin{aligned}
4\,|\,abc\,|^2\;|\,def\,|^2 &= R+(adbecfa)+(afbdcea)+(aebfcda) \\
&\qquad +(adbfcea)+(afbecda)+(aebdcfa) \\
&= R+[a-b-c-a], \text{ say};
\end{aligned}$$

then $\qquad [a-b-c-a] = R.$

This last is the sum of two equations, for, if S be the sum of the sixty

possible forms $(ABCDEF)\ I_6$,

$$[ad,\ be,\ c,\ f,\ a]+[af,\ bd,\ c,\ e,\ a]+[ae,\ bf,\ c,\ d,\ a]$$
$$+[a,\ d,\ be,\ cf,\ a]+[a,f,\ bd,\ ce,\ a]+[a,\ e,\ bf,\ cd,\ a]$$
$$+[ad,\ b,\ e,\ cf,\ a]+[af,\ b,\ d,\ ce,\ a]+[ae,\ b,f,\ cd,\ a]$$
$$+[abc\ \dots\ a]+[acb\ \dots\ a]+[ab\ \dots\ ca]$$
$$-S+[a-b-c-a]$$
$$=6\left[(adbecfa)+(afbdcea)+(aebfcda)\right]=R, \qquad \text{VI.}$$

since S is reducible, it being the sum of the ten expressions $[a-b-c-a]$.

The results of operating with $[ef]\left[f_0\frac{\partial}{\partial e_0}+f_1\frac{\partial}{\partial e_1}+f_2\frac{\partial}{\partial e_2}\right]$ on $[ea]$, $[ee]$ and $(abcdea)$ are $[ef][fa]$, $[ef]^2$, and $(abcdefa)+(abcdfea)$ respectively. Hence from any relation $\Sigma I_5=R$ there may be at once deduced one of the form $\Sigma I_6=R$. The above operation will be found equivalent to substituting the coefficients of that covariant of two quartics which is of the same type as the Hessian for the coefficients of a single quartic.

Applying this operator to V., we obtain

$$[ef]\left[f_0\frac{\partial}{\partial e_0}+f_1\frac{\partial}{\partial e_1}+f_2\frac{\partial}{\partial e_2}\right]\ |\ ABCDE\ |$$
$$=|\ ABCD,\ EF\ |$$
$$=(abdcefa)+(abdcfea)-(acbefda)-(acbfeda)+R$$
$$=(abefdca)+(abfedca)-(adbcefa)-(adbcfea)+R$$
$$=\tfrac{1}{2}\left\{-(afdbeca)-(aedbfca)+(afbdeca)+(aebdfca)\right\}+R,$$

using equations of the form VI.

$$\text{Hence}\quad (abdcefa)+(abdcfea)+(adbcefa)+(adbcfea)+R$$
$$=(abefdca)+(abfedca)+(acbefda)+(acbfeda)+R.$$

If b and e, f and d be interchanged in this, the first line is unaltered; it therefore

$$=(aebdfca)+(aedbfca)+(acebdfa)+(acedbfa)+R=R,$$

as is seen by adding the three lines and using equations of the

form VI.; therefore

$$\begin{aligned}
| ABCD, EF | \\
&= \tfrac{1}{2}\left\{-(afdbeca)-(aedbfca)+(afbdeca)+(aebdfca)\right\}+R \\
&= -(afdbeca)-(aedbfca)+R = (afbdeca)+(aebdfca)+R \\
&= | ABCD, FE | .
\end{aligned}$$

It follows that the type I_6 may be expressed in the determinant forms $| ABCD, EF |$; for

$$\begin{aligned}
(afdbeca)+(aedbfca) &= | ABDC, EF | + R, \\
-(aedbfca)-(aecbfda) &= | EFAB, DC | + R, \\
(aecbfda)+(afdbeca) &= | DCEF, AB | + R.
\end{aligned}$$

Therefore

$$2(afdbeca) = | ABDC, EF | + | EFAB, DC | + | DCEF, AB | + R, \qquad \text{VII.}$$

There are fifteen possible forms $| ABCD, EF |$. Written in full,

$$| ABCD, EF |$$

$$= 2 \begin{vmatrix} A_0 & B_0 & C_0 & D_0 & E_0F_2-2E_1F_1+E_2F_0 \\ A_1 & B_1 & C_1 & D_1 & \tfrac{1}{2}(E_0F_3-E_1F_2-E_2F_1+E_3F_0) \\ A_2 & B_2 & C_2 & D_2 & \tfrac{1}{6}(E_0F_4+2E_1F_3-6E_2F_2+2E_3F_1+E_4F_0) \\ A_3 & B_3 & C_3 & D_3 & \tfrac{1}{2}(E_1F_4-E_2F_3-E_3F_2+E_4F_1) \\ A_4 & B_4 & C_4 & D_4 & E_2F_4-2E_3F_3+E_4F_2 \end{vmatrix}$$

Amongst these forms one kind of equation exists, viz.,

$$| ABCD, EF | + | EABC, DF | + | DEAB, CF | + | CDEA, BF | + | BCDE, AF | = 0, \qquad \text{VIII.}$$

as may be verified by taking the coefficients of $F_0, F_1, \ldots, F_4$ in turn. When $(abcdefa)$ is expressed in the determinant forms, $[abc \ldots a] = R$ is identically satisfied, and $[ab, cd, e, f, a] = R$ becomes the sum of two equations VIII. Hence there are no relations between the forms $| ABCD, EF |$ beyond those included in VIII. There are six relations VIII., and five are independent. Hence there are ten

independent irreducible forms $(ABCDEF)\ I_6$. For a system of N quartics there are

$$10\binom{N}{6}+20\binom{N}{5}+10\binom{N}{4}=10\binom{N+2}{6}$$

irreducible invariants of this type.

Invariant types of higher degree than 6 are reducible. For, using the operator $[fg]\left[g_0\frac{\partial}{\partial f_0}+g_1\frac{\partial}{\partial f_1}+g_2\frac{\partial}{\partial f_2}\right]$, we obtain from VII.

$$2\,(afgdbeca)+2\,(agfdbeca)$$
$$=|\,A, B, D, C, EFG\,|+|\,E, FG, A, B, DC\,|+|\,D, C, E, FG, AB\,|. \qquad \text{IX.}$$

Therefore

$$R=[a\ldots beca]$$
$$=|\,A,B,D,C,EFG\,|+|\,E,FG,A,B,DC\,|+|\,D,C,E,FG,AB\,|$$
$$+|\,A,B,G,C,EDF\,|+|\,E,DF,A,B,GC\,|+|\,G,C,E,DF,AB\,|$$
$$+|\,A,B,F,C,EGD\,|+|\,E,GD,A,B,FC\,|+|\,F,C,E,GD,AB\,|.$$

Interchange A and B, and add the result to the original equation

$$|\,D, C, E, FG, AB\,|+|\,G, C, E, DF, AB\,|+|\,F, C, E, GD, AB\,|=R. \qquad \text{X.}$$

Hence also

$$|\,A, B, D, C, EFG\,|+|\,A, B, G, C, EDF\,|+|\,A, B, F, C, EGD\,|$$
$$+|\,E, FG, A, B, DC\,|+|\,E, DF, A, B, GC\,|+|\,E, GD, A, B, FC\,|$$
$$=R. \qquad \text{XI.}$$

But, from VIII.,

$$|\,A, B, D, C, EFG\,|+|\,FG, A, B, D, EC\,|+|\,C, FG, A, B, ED\,|$$
$$+|\,D, C, FG, A, EB\,|+|\,B, D, C, FG, EA\,|=R.$$

Hence, using equations of the form X.,

$$|\,A, B, D, C, EFG\,|+|\,A, B, G, C, EDF\,|+|\,A,B,F,C,EGD\,|$$
$$=-|\,C, FG, A, B, ED\,|-|\,C, DF, A, B, EG\,|-|\,C, DG, A, B, EF\,|$$
$$+R;$$

and therefore XI. becomes

$$| E, FG, A, B, DC | + | E, DF, A, B, GC | + | E, GD, A, B, FC |$$
$$= | C, FG, A, B, ED | + | C, DF, A, B, EG | + | C, DG, A, B, EF | + R$$
$$= | C, FG, E, B, AD | + | C, DF, E, B, AG | + | C, GD, E, B, AF | + R$$
$$= | A, FG, E, B, CD | + | A, DF, E, B, CG | + | A, GD, E, B, CF | + R$$
$$= R,$$

the last two equations being obtained from the first by substitutions. Interchange F and G, C and D in the last of these, and add the result to the original equation; then

$$2 \,| A, FG, E, B, CD | = R,$$

and, in virtue of the equation obtained from VIII.,

$$| A, B, C, D, EFG | = R.$$

Therefore IX. becomes

$$(afgdbeca) + (agfdbeca) = R.$$

From this it may at once be deduced that the sum of $(abcdefga)$ and any form obtained from it by a substitution formed of an odd number of transpositions is reducible.

Hence $(abcdefga) + (agfedcba) = R$ or $2\,(abcdefga) = R$,

and the type I_7 is reducible.

Operate on $(abcdefga)$ with $[gh]\left[h_0\dfrac{\partial}{\partial g_0} + h_1\dfrac{\partial}{\partial g_1} + h_2\dfrac{\partial}{\partial g_2}\right]$; then

$$(abcdefgha) + (abcdefhga) = R. \qquad \text{XII.}$$

The equation $\qquad [ab, cd, ef, gh, a] = R$

has six terms, each obtainable from the first by means of a substitution formed of an even number of transpositions; therefore, by repeated use of XII., it gives

$$6\,(abcdefgha) = R.$$

The reduction of invariant types of higher degree follows in the same way.

4. The equations for covariants order 2 are:

(i.) Those obtained from the factors $[ab]$ only, viz.,

$$[a \ldots e] = R, \quad [a, bc, d, e, f] = R, \ \&c.$$

(ii.) Those obtained from III. by writing in that identity $h_1x_1^2+h_2x_1x_2$ for h_2, $h_0x_1^2-h_2x_2^2$ for $2h_1$, and $-h_0x_1x_2-h_1x_2^2$ for h_0, and therefore $|\,ahx^2\,|$ for $[ah]$; these are of the forms

$$[a\,...]=R, \quad [a,\, bc,\, d,\, e,]=R, \text{ \&c.}$$

(iii.) A system of equations obtained from II., thus:

$$\begin{aligned}[ef]\,|\,abc\,|\,d_{x^2} = d_{x^2}\,\{[af]\,|\,bce\,| + [bf]\,|\,cae\,| + [cf]\,|\,abe\,|\,\} \\ = [af]\,\{[bd]\,|\,cex^2\,| + [cd]\,|\,ebx^2\,| + [ed]\,|\,bcx^2\,|\,\} \\ +[bf]\,\{[cd]\,|\,aex^2\,| + [ad]\,|\,ecx^2\,| + [ed]\,|\,cax^2\,|\,\} \\ +[cf]\,\{[ad]\,|\,bex^2\,| + [bd]\,|\,eax^2\,| + [ed]\,|\,abx^2\,|\,\}.\end{aligned}$$

Multiply this identity by $[ef][bd][ca]$; then

$$\begin{aligned}[bd,\, ca,\, fe] \equiv (bdcafe)+(cafedb)+(efbdca)+(cadbfe)+(efcadb) \\ +(bdefca) = R.\end{aligned}$$

Similarly the following may be deduced:—

$$\begin{aligned}[bd,\, ca,\, e] \equiv (bdcae)+(caedb)+(ebdca)+(cadbe)+(ecadb) \\ +(bdeca) \quad = R,\end{aligned}$$

$$[b,\, ca,\, e] \equiv (bcae)+(caeb)+(ebca)+(cabe)+(ecab)+(beca)=R.$$

For the types ${}_2C_2$, ${}_2C_3$ there are no equations; and for a system of N quartics there are $\binom{N}{2}$ and $2\binom{N+1}{3}+\binom{N}{3}$ irreducible covariants of these two types respectively.

There are six equations like

$$[b,\, ca,\, d]=R$$

amongst the twelve forms $(ABCD)\,{}_2C_4$, all of which are independent; the equations

$$[a\,...]=R$$

are, however, deducible from these. Hence there are six independent irreducible forms $(ABCD)\,{}_2C_4$; and for N quartics there are $3\binom{N+2}{4}+3\binom{N+1}{4}$ irreducible covariants of this type.

As regards higher degrees, consider the form obtained by operating with $\epsilon_x\left[x_1\dfrac{\partial}{\partial\epsilon_2}-x_2\dfrac{\partial}{\partial\epsilon_1}\right]$ on $|\,ABCDE\,|$; where $\epsilon_x^4\equiv e_{x^2}^2$; from V.,

$$|\,ABCDE\,| = R+(\alpha\beta)^2(\beta\delta)^2(\delta\gamma)^2(\gamma\epsilon)^2(\epsilon\alpha)^2-(\alpha\gamma)^2(\gamma\beta)^2(\beta\epsilon)^2(\epsilon\delta)^2(\delta\alpha)^2;$$

therefore

$$\tfrac{1}{2}\epsilon_x\left[x_1\frac{\partial}{\partial\epsilon_1}-x_2\frac{\partial}{\partial\epsilon_1}\right] \mid ABCDE \mid \equiv \tfrac{1}{2} \mid ABCD,\ Ex^2 \mid \equiv \tfrac{1}{2}\Delta_E \text{ (say)}$$

$$= R+(\alpha\beta)^2(\beta\delta)^2(\delta\gamma)^2(\gamma\epsilon)(\epsilon\alpha)\left\{\gamma_x(\epsilon\alpha)-\alpha_x(\gamma\epsilon)\right\}\epsilon_x$$

$$-(\alpha\gamma)^2(\gamma\beta)^2(\beta\epsilon)(\epsilon\delta)(\delta\alpha)^2\left\{\beta_x(\epsilon\delta)-\delta_x(\beta\epsilon)\right\}\epsilon_x$$

$$= R+(eabdc)-(abdce)-(edacb)+(dacbe) \equiv R+[eabdc] \text{ (say)}$$

$$= R+[ebdac] = R+[eacbd] = R+[ecbad] = R+[eadcb]$$

$$= R+[edcab]. \qquad \text{XIII.}$$

Now $\quad [dc,\ ba,\ e]+[ab,\ cd,\ e]+[a,\ bed,\ c]+[b,\ cea,\ d]$

$$= [bacde]-[dcabe]-[abdce]+[cdbae]+3(cbeda)+3(bcead).$$

Therefore $\quad (cbeda)+(bcead)$

$$= \tfrac{1}{6}\left\{ \mid BCDE,\ Ax^2 \mid + \mid EABC,\ Dx^2 \mid - \mid CDEA,\ Bx^2 \mid - \mid DEAB,\ Cx^2 \mid \right\}+R$$

$$= \tfrac{1}{6}\left\{\Delta_A-\Delta_B-\Delta_C+\Delta_D\right\}+R = \Sigma\Delta+R. \qquad \text{XIV.}$$

Again,

$$[cd,ab,e]-[b,aed,c]+[a,bed,c]-[cdabe]+[badce]-[abdce]+[cdbae]$$

$$= 3(caedb)-3(cbeda)+(aedcb)+(cbaed)-(cabed)-(bedca)-(cebad)$$

$$-(aecdb)+(becda)+(ceabd)$$

$$= 3(caedb)-3(cbeda)+(cbdae)+(edacb)-(edbca)-(cadbe)-(adbce)$$

$$-(dbcae)+(dacbe)+(bdace)+R+\Sigma\Delta \text{ (by XIV.)}$$

$$= 5(caedb)-5(cbeda)+[ad,\ cb,\ e]+[bc,\ da,\ e]-[ac,\ db,\ e]$$

$$-[bd,\ ca,\ e]+R+\Sigma\Delta.$$

Therefore $\quad (caedb)-(cbeda) = R+\Sigma\Delta. \qquad$ XV.

Again, using XV.,

$$[acedb]+[bdeca]-[e\ \ldots\ a]$$

$$= (ecdab)+(edacb)-(edbca)-(ecdba)+R+\Sigma\Delta.$$

Subtract the sum of the results of interchanging c and a, c and b, respectively in this equation from its original form; then, with the help of XV., we obtain

$$(edacb)+(edcba)+(edbac)-(edcab)-(edabc)-(edbac) = R+\Sigma\Delta.$$

But $\quad [ed\ \ldots] = R;$

therefore $\quad (edacb)+(edcba)+(edbac) = R+\Sigma\Delta. \qquad$ XVI.

The equations XV. and XVI. contain all the equations for $(abcde)$. It follows from XV. and XVI. that any form

$$ {}_2C_5 = \Sigma\,(a\,\ldots) + \Sigma\Delta + R\,; $$

further they prove that forms having a definite letter b in the second or fifth place can be expressed in terms of those having b in the third or fourth place, and at the same time a in the first place; XV. leaves only six of these forms to be discussed, and amongst these we have one relation, viz.,

$$ (acbde) + (acebd) + (aebcd) + (adcbe) + (adbec) + (aedbc) = R + \Sigma\Delta. $$

Hence there are five independent irreducible forms $(abcde)$, and five forms $|\,ABCD,\,Ex^2\,|$ which have yet to be discussed. These determinant forms prove to be reducible. Thus

$$ [abcde] + [bcdea] + [cdeab] + [deabc] + [eabcd] \equiv 0\,; $$

therefore $$ \Delta_A + \Delta_B + \Delta_C + \Delta_D + D_E = R. \qquad \text{XVII.} $$

Now, from XVI., XV., and $[e,\, da,\, c,\, b,] = R$ we obtain

$$ \begin{aligned} (edacb) + (ecdab) + (ebcda) &= R - (edabc) - (ebdac) - (ecbda) \\ &= \Sigma\Delta + R = \lambda_1\Delta_A + \lambda_2\Delta_B + \lambda_3\Delta_C + \lambda_4\Delta_D + \lambda_5\Delta_E + R \text{ (say)} \\ &= \lambda_1\Delta_A + \lambda_3\Delta_B + \lambda_2\Delta_C + \lambda_4\Delta_D + \lambda_5\Delta_E + R, \end{aligned} $$

since the interchange of B and C merely changes the sign of each expression. Hence $\lambda_2 = \lambda_3$, and, in virtue of XVII., they may be each taken to be zero. Interchange e and b, a and d in the above result; then

$$ (badce) + (bcade) + (becad) = \lambda_4\Delta_A + \lambda_5\Delta_B + \lambda_1\Delta_D + R\,; \qquad \text{XVIII.} $$

adding,

$$ \begin{aligned} (ebcda) + (becad) &= (\lambda_1 + \lambda_4)(\Delta_A + \Delta_D) + \lambda_5(\Delta_B + \Delta_E) + R \\ &= \tfrac{1}{6}\left\{-\Delta_A - \Delta_D + \Delta_B + \Delta_E\right\} + R, \end{aligned} $$

in virtue of XIV. Therefore

$$ \lambda_5 = \tfrac{1}{6} = -(\lambda_1 + \lambda_4). $$

Substituting symmetrically the results of XVIII.,

$$ \begin{aligned} 9\Delta_E &= 3\,[eabdc] + 3\,[ebdac] + 3\,[eacbd] + 3\,[ecbad] + 3\,[eadcb] \\ &\qquad + 3\,[edcab] + R \\ &= 2\left\{-12\lambda_5\Delta_E - 3\,(\lambda_1 + \lambda_4)(\Delta_A + \Delta_B + \Delta_C + \Delta_D)\right\} + R = -5\Delta_E + R. \end{aligned} $$

Therefore these determinant forms are reducible.

Hence for N quartics there are

$$5\binom{N}{5}+12\binom{N}{4}+9\binom{N}{3}+2\binom{N}{2}=2\binom{N+3}{5}+3\binom{N+2}{5}$$

independent irreducible covariants of the type ${}_2C_5$.

All covariants of order 2 and of degree higher than 5 are reducible; for

$$(abcdefa) = \Sigma \mid ABCD,\, EF \mid +R.$$

Operate with $a_x\left[x_1\dfrac{\partial}{\partial a_2}-x_2\dfrac{\partial}{\partial a_1}\right]$; then

$$(abcdef)-(bcdefa) = R+\Sigma\Delta.$$

The determinant $\mid Ax^2,\, B,\, C,\, D,\, EF \mid$ is obtainable from $\mid Ax^2, B, C, D, E \mid$ by an operation $[ef]\left[f_0\dfrac{\partial}{\partial e_0}+f_1\dfrac{\partial}{\partial e_1}+f_2\dfrac{\partial}{\partial e_2}\right]$, and is therefore reducible.

Further, $a_x\left[x_1\dfrac{\partial}{\partial a_2}-x_2\dfrac{\partial}{\partial a_1}\right] \mid BCDE,\, AF \mid$ is reducible owing to VIII. Therefore

$$(abcdef)-(bcdefa) = R.$$

Now
$$[a,\, bcde,\, f] = R;$$

therefore
$$3\,(abcdef)+3\,(fbcdea) = R.$$

Hence the sum of $(abcdef)$ and a form obtained from it by an odd substitution is reducible; therefore

$$(abcdef)+(bcdefa) = R,$$

and hence $(abcdef)$ is reducible.

The reducibility of forms of higher degree and of the second order may be obtained in exactly the same way as it was for invariants.

5. Covariants of order higher than 2 may be obtained from those of order 4 lower, by writing, in the quadratic symbolical expressions for these, for a_0, x_2^2; for a_1, $-x_1x_2$; for a_2, x_1^2. It is at once apparent that all covariants of order higher than 6 are reducible; and that the only forms which have yet to be discussed are ${}_4C_1$, ${}_4C_2$, ${}_4C_3$, ${}_4C_4$, ${}_4C_5$, ${}_6C_2$, ${}_6C_3$, ${}_6C_4$.

For the forms order 4, the only further reductions possible are those due to products of two covariants order 2. The first form to be affected by these is ${}_4C_4$.

Take then such a product

$$2\,[ab]\mid abx^2\mid[cd]\mid cdx^2\mid$$
$$=[ab][cd]\begin{vmatrix}[ac] & [ad] & a_{x^2}\\ [bc] & [bd] & b_{x^2}\\ c_{x^2} & d_{x^2} & 0\end{vmatrix}$$
$$=-(abx^2dca)+(abx^2cda)+(abcdx^2a)-(abdcx^2a).$$

But, from V., $(abx^2dca)=(adbcx^2a)+\mid ABCDx^4\mid+R,$

$$(abdcx^2a)=(acbx^2da)+\mid ABCDx^4\mid+R.$$

Therefore

$$2\mid ABCDx^4\mid=(abx^2cda)+(abcdx^2a)-(adbcx^2a)-(acbx^2da)+R.$$

Perform the substitutions (bcd) and (bdc), and add the results; then

$$6\mid ABCDx^4\mid=R.$$

There are then only six forms to discuss, connected by the equations

$$\begin{aligned}&(abx^2cda)+(abcdx^2a)+R\\ &\quad=(adbcx^2a)+(adx^2bca)+R=(acx^2dba)+(acdbx^2a)+R\\ &\quad=\tfrac{1}{3}\{(abx^2cda)+(abcdx^2a)+(adbcx^2a)+(adx^2bca)+(acx^2dba)\\ &\qquad\qquad+(acdbx^2a)\}+R\\ &\quad=R.\end{aligned}$$

Hence there are three independent forms $(ABCD)\,{}_4C_4$, and for N quartics there are $3\binom{N+1}{4}$ independent irreducible covariants of the type ${}_4C_4$.

Since I_6 is expressible in the forms $\mid ABCD,\,EF\mid$, which are connected by equations VIII., it follows that ${}_4C_5$ may be expressed in terms of the determinants $\mid ABCx^4,\,EF\mid$; which are reducible, since $\mid ABCDx^4\mid$ is reducible.

Hence the type ${}_4C_5$ is reducible.

As regards the types degree 6, there are $\binom{N}{2}$ irreducible forms of the type ${}_6C_2$ for N quartics. For ${}_6C_3$ it is necessary to refer to ${}_2C_4$; the equations for ${}_2C_4$ give

$$(cax^2b)+(bx^2ca)=R\quad\text{or}\quad(bx^2ca)=R+(bx^2ac)$$

and $$(bcx^2a)+(cx^2ab)+(cx^2ba)+(acx^2b)=R.$$

Hence $$2\,(cx^2ab)=(ax^2cb)+(bx^2ca)+R,$$

and therefore the difference of any two forms $(ABC)\,{}_6C_3$ is reducible.

Hence the number of independent irreducible covariants of the type ${}_6C_3$ for N quartics is $\binom{N+2}{3}$.

Reference to the work for ${}_2C_5$ shows that forms ${}_6C_4 = \Sigma\,(x^2 \ldots) + R$, but $(x^2 \ldots) = 0$; hence the type ${}_6C_4$ is reducible.

6. The complete system of irreducible concomitants for a set of N quartics is as follows:—

Type.		Number of Independent Forms.
I_2	....	$\binom{N+1}{2}$
I_3	....	$\binom{N+2}{3}$
I_4	....	$\binom{N+1}{4}+\binom{N+2}{4}$
I_5	....	$\binom{N}{5}+2\binom{N+1}{5}+3\binom{N+2}{5}$
I_6	....	$10\binom{N+2}{6}$
${}_2C_2$	....	$\binom{N}{2}$
${}_2C_3$	....	$\binom{N}{3}+2\binom{N+1}{3}$
${}_2C_4$	....	$3\binom{N+1}{4}+3\binom{N+2}{4}$
${}_2C_5$	....	$3\binom{N+2}{5}+2\binom{N+3}{5}$
${}_4C_1$	....	N
${}_4C_2$	....	$\binom{N+1}{2}$
${}_4C_3$	....	$2\binom{N+1}{3}$
${}_4C_4$	....	$3\binom{N+1}{4}$
${}_6C_2$	....	$\binom{N}{2}$
${}_6C_3$	....	$\binom{N+2}{3}$.

The Scattering of Electric Waves by a Dielectric Sphere. *By* A. E. H. LOVE. Read February 9th, 1899. Received February 16th, 1899.

1. The problem of the diffraction of light by small particles of spherical shape has been considered, from the point of view of the electro-magnetic theory of light, by Lord Rayleigh, in the *Philosophical Magazine* for August, 1881. An approximate solution of the problem is there given, which depends on the two suppositions: (1) that the dielectric constants for the material of the sphere and the external medium are very nearly equal, (2) that the radius of the sphere is very small compared with the wave-length of the incident light. The most important results are that, if terms of the lowest order that occur are alone retained, the waves scattered in any direction perpendicular to the direction of propagation of the incident light are completely polarized, that this result still holds good if the difference of dielectric constants is not small, and that when a second approximation is made the direction in which the scattered wave is most nearly polarized makes a slightly obtuse angle with the direction of propagation of the incident waves.

It seemed to me that it might be not without interest to work out a complete solution of the problem when the incident waves are plane, the sphere is of any size, and the difference of the dielectric constants of the internal and external medium is any given number. We should expect such a complete solution to verify exactly Lord Rayleigh's approximate result for very small spheres when terms of the lowest order only are retained, and to point to the same kind of conclusions for somewhat larger spheres when a second approximation is worked out.

2. The analysis requisite for such a complete solution has been developed by Prof. Lamb in a series of papers in the *Proceedings* (Vols. XIII. and XV.), and it will only be necessary here to make a brief statement of the equations and the types of solution employed.

In a dielectric medium with dielectric constant K and magnetic permeability μ, the equations satisfied by the electric force (X, Y, Z)

and the magnetic force (α, β, γ) are

$$\left.\begin{aligned} \frac{\mathrm{K}}{c}\frac{\partial}{\partial t}(X, Y, Z) &= \operatorname{curl}(\alpha, \beta, \gamma), \\ -\frac{\mu}{c}\frac{\partial}{\partial t}(\alpha, \beta, \gamma) &= \operatorname{curl}(X, Y, Z), \end{aligned}\right\} \tag{1}$$

wherein by the curl of a vector (u, v, w) is meant the vector whose resolved parts parallel to the axes are

$$\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y},$$

and the constant c is the velocity of light in free ether.

The above equations, when (X, Y, Z) and (α, β, γ) are proportional to the same simple harmonic function of the time $e^{\iota\kappa_1 ct}$, show that the resolved parts of these vectors are related solutions of the same system of equations

$$\left.\begin{aligned} (\nabla^2+\kappa^2)\,u = 0, \quad (\nabla^2+\kappa^2)\,v = 0, \quad (\nabla^2+\kappa^2)\,w = 0, \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0, \end{aligned}\right\} \tag{2}$$

where $$\kappa^2 = \kappa_1^2\mathrm{K}\mu.$$

The solutions of these equations which involve spherical surface harmonics, and are finite at the origin, fall into two types. The first type is given by

$$(u, v, w) = \psi_n(\kappa r)\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}, \quad z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}, \quad x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\omega_n, \tag{3}$$

where ω_n is a spherical solid harmonic of positive degree n, and

$$\psi_n(\eta) = (-)^n\, 1.3.5 \ldots (2n+1)\left(\frac{1}{\eta}\frac{d}{d\eta}\right)^n \frac{\sin\eta}{\eta}; \tag{4}$$

the numerical coefficient is so chosen that

$$\mathrm{Lt}_{\eta=0}\,\psi_n(\eta) = 1.$$

Since curl (u, v, w) satisfies the same equations as (u, v, w), we obtain the second type. In fact, the resolved part parallel to x of curl (u, v, w), where u, v, w are as above, is

$$-(n+1)\,\psi_{n-1}(\kappa r)\frac{\partial\omega_n}{\partial x}+\frac{n\kappa^2 r^{2n+3}}{(2n+1)(2n+3)}\psi_{n+1}(\kappa r)\frac{\partial}{\partial x}\frac{\omega_n}{r^{2n+1}}, \tag{5}$$

and the resolved parts parallel to y and z can be put down by writing $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ for $\frac{\partial}{\partial x}$.

The functions $\psi_n(\eta)$ satisfy certain sequence equations which can be written

$$\eta \frac{d}{d\eta} \psi_n(\eta) = -\frac{\eta^2}{2n+3} \psi_{n+1}(\eta) = (2n+1)\{\psi_{n-1}(\eta) - \psi_n(\eta)\}. \quad (6)$$

The two types of solutions which correspond to waves propagated outwards are of the same form as those which correspond to disturbances which are finite at the origin, and are obtained from them by writing everywhere $E_n(\kappa r)$ for $\psi_n(\kappa r)$, where

$$E_n(\eta) = (-)^n 1.3.5 \ldots (2n+1) \left(\frac{1}{\eta} \frac{d}{d\eta}\right)^n \frac{e^{-\iota\eta}}{\eta}, \quad (7)$$

and the functions $E_n(\eta)$ satisfy the same sequence equations as the functions $\psi_n(\eta)$.

With a view to the satisfaction of conditions at a spherical surface, we require the radial components of the above vectors. The vector given by the solutions of the first type is purely transverse. The radial component of the vector given by the solutions of the second type written as above (5) is easily seen to be

$$-\frac{1}{r} n(n+1) \psi_n(\kappa r) \omega_n. \quad (8)$$

If, then, we add to the components given by (5) and the similar forms the quantities such as

$$xr^{-2} n(n+1) \psi_n(\kappa r) \omega_n,$$

we shall have the parts contributed to these components by the transverse components of the vector, and after a little reduction we obtain forms of which the type is

$$\left\{\frac{n}{2n+1} \psi_n(\kappa r) - \psi_{n-1}(\kappa r)\right\} \left\{(n+1) \frac{\partial \omega_n}{\partial x} + nr^{2n+1} \frac{\partial}{\partial x} \frac{\omega_n}{r^{2n+1}}\right\}. \quad (9)$$

This is the part of the x-component of curl (u, v, w) contributed by the transverse components of curl (u, v, w).

The corresponding expressions in the case of waves propagated outwards are of the same forms with $E_n(\kappa r)$ in place of $\psi_n(\kappa r)$.

3. We shall now suppose that the space outside a sphere of radius R is occupied by free ether for which K and μ are both unity,

and that there is incident upon the sphere a train of plane polarized waves propagated in the negative direction of the axis of z. In these incident waves we shall suppose the electric force is parallel to y, and is given by

$$Y = A e^{\iota\kappa(ct+z)}, \tag{10}$$

and that the magnetic force is parallel to x, and is given by

$$a = A e^{\iota\kappa(ct+z)}; \tag{11}$$

these forms are compatible with the equations (1) when **K** and μ are each unity. With a view to the satisfaction of the boundary conditions, we wish to have these forces expressed by means of solutions of the first type and solutions of the second type which are finite at the origin. The first step is to expand $e^{\iota\kappa z}$ in terms of surface harmonics. This expansion is known to be

$$e^{\iota\kappa z} = 1 + \sum_{n=1}^{n=\infty} \frac{(\iota\kappa r)^n}{1.3.5 \ldots (2n-1)} \psi_n(\kappa r) P_n\left(\frac{z}{r}\right). \tag{12}$$

Now consider electric and magnetic forces given by

$$\left.\begin{aligned} X = 0, \quad Y = A e^{\iota\kappa ct} \sum_0^\infty \psi_n(\kappa r) V_n, \quad Z = 0, \\ a = A e^{\iota\kappa ct} \sum_0^\infty \psi_n(\kappa r) V_n, \quad \beta = 0, \quad \gamma = 0, \end{aligned}\right\} \tag{13}$$

where V_n is a spherical solid harmonic of degree n, $V_0 = 1$, and, for $n > 1$,

$$V_n = \frac{(\iota\kappa r)^n}{1.3.5 \ldots (2n-1)} P_n\left(\frac{z}{r}\right). \tag{14}$$

We can arrange the expressions for these forces as sums of solutions of the first type and solutions of the second type. Let the solutions of the first type that occur in the expression of the electric force be formed with solid harmonics of which the one of degree n is φ_n, and let the solutions of the first type that occur in the expression of the magnetic force be formed with solid harmonics of which the one of degree n is χ_n. Then the complete expressions for X, Y, Z are of the forms

$$\begin{aligned} X = \Sigma \psi_n(\kappa r) \left(y \frac{\partial \phi_n}{\partial z} - z \frac{\partial \varphi_n}{\partial y}\right) \\ + \Sigma \frac{1}{\iota\kappa} \left[-(n+1) \psi_{n-1}(\kappa r) \frac{\partial \chi_n}{\partial x} + \frac{n\kappa^2 r^{2n+3}}{(2n+1)(2n+3)} \psi_{n+1}(\kappa r) \frac{\partial}{\partial x} \frac{\chi_n}{r^{2n+1}}\right], \end{aligned} \tag{15}$$

where the summation refers to the different orders of harmonics. In like manner the complete expressions for α, β, γ are of the forms

$$\alpha = \Sigma \psi_n(\kappa r)\left(y\frac{\partial \chi_n}{\partial z} - z\frac{\partial \chi_n}{\partial y}\right)$$

$$-\Sigma\frac{1}{\iota\kappa}\left[-(n+1)\psi_{n-1}(\kappa r)\frac{\partial \phi_n}{\partial x} + \frac{n\kappa^2 r^{2n+3}}{(2n+1)(2n+3)}\psi_{n+1}(\kappa r)\frac{\partial}{\partial x}\frac{\phi_n}{r^{2n+1}}\right]. \qquad (16)$$

The normal components of (X, Y, Z) and (α, β, γ) at a sphere $r = R$ are respectively

$$\Sigma - \frac{n(n+1)}{\iota\kappa R}\psi_n(\kappa R)\chi_n \quad \text{and} \quad \Sigma\frac{n(n+1)}{\iota\kappa R}\psi_n(\kappa R)\phi_n,$$

in which ϕ_n and χ_n are to have their values for $r = R$. Now we know* that there cannot be two different expressions in terms of solutions of the first and second type which yield the same radial components of both electric and magnetic force at a spherical surface, and are both finite within the surface; we therefore obtain the result that, when $r = R$,

$$\left.\begin{aligned}\Sigma - \frac{n(n+1)}{\iota\kappa r}\psi_n(\kappa r)\chi_n &= Ae^{\iota\kappa ct}\sum_0^\infty \psi_n(\kappa r)\frac{y}{r}V_n,\\ \Sigma\ \frac{n(n+1)}{\iota\kappa r}\psi_n(\kappa r)\phi_n &= Ae^{\iota\kappa ct}\sum_0^\infty \psi_n(\kappa r)\frac{x}{r}V_n,\end{aligned}\right\} \qquad (17)$$

and, by equating surface harmonics of the same order at $r = R$, we find

$$\left.\begin{aligned}\phi_n &= \frac{A\iota\kappa e^{\iota\kappa ct}}{n(n+1)\psi_n(\kappa R)}\left[\frac{R^2}{2n+1}\psi_{n+1}(\kappa R)\frac{\partial V_{n+1}}{\partial x} - \frac{r^{2n+1}}{2n-1}\psi_{n-1}(\kappa R)\frac{\partial}{\partial x}\left(\frac{V_{n-1}}{r^{2n-1}}\right)\right],\\ \chi_n &= -\frac{A\iota\kappa e^{\iota\kappa ct}}{n(n+1)\psi_n(\kappa R)}\left[\frac{R^2}{2n+1}\psi_{n+1}(\kappa R)\frac{\partial V_{n+1}}{\partial y} - \frac{r^{2n+1}}{2n-1}\psi_{n-1}(\kappa R)\frac{\partial}{\partial y}\left(\frac{V_{n-1}}{r^{2n-1}}\right)\right].\end{aligned}\right\} \qquad (18)$$

4. The expression in the required form of the electric and magnetic forces in the incident wave has just been effected. We shall now suppose that the forces in the scattered wave are expressed by similar forms with ϕ_n' and χ_n' in place of ϕ_n and χ_n, and $E_n(\kappa r)$ instead

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* Lamb, *Phil. Trans.*, Part 2, 1883, p. 533.

of $\psi_n(\kappa r)$. We know that this is the complete expression for a system of waves propagated outwards.

The material of the obstructing sphere will be taken to be of dielectric constant K and magnetic permeability μ. We shall suppose the disturbance within the sphere to be derived from solid harmonics ϕ_n'' and χ_n''. The disturbance being proportional to $e^{\iota\kappa ct}$, the components of the forces now satisfy such equations as

$$(\nabla^2+\kappa'^2)\chi = 0,$$

where
$$\kappa'^2 = \kappa^2 \mathrm{K}\mu, \tag{19}$$

and we accordingly take, as the type of X, Y, Z,

$$X = \Sigma\psi_n(\kappa' r)\left(y\frac{\partial\phi_n''}{\partial z} - z\frac{\partial\phi_n''}{\partial y}\right)$$
$$+\frac{1}{\iota\kappa\mathrm{K}}\Sigma\left[-(n+1)\psi_{n-1}(\kappa' r)\frac{\partial\chi_n''}{\partial x}+\frac{n\kappa'^2 r^{2n+3}}{(2n+1)(2n+3)}\psi_{n+1}(\kappa' r)\frac{\partial}{\partial x}\left(\frac{\chi_n''}{r^{2n+1}}\right)\right], \tag{20}$$

and, as the type of α, β, γ,

$$\alpha = \Sigma\psi_n(\kappa' r)\left(y\frac{\partial\chi_n''}{\partial z} - z\frac{\partial\chi_n''}{\partial y}\right)$$
$$-\frac{1}{\iota\kappa\mu}\Sigma\left[-(n+1)\psi_{n-1}(\kappa' r)\frac{\partial\phi_n''}{\partial x}+\frac{n\kappa'^2 r^{2n+3}}{(2n+1)(2n+3)}\psi_{n+1}(\kappa' r)\frac{\partial}{\partial x}\left(\frac{\phi_n''}{r^{2n+1}}\right)\right]. \tag{21}$$

Then utilizing the expressions (9) for the contributions of such solutions as we have obtained to the tangential components of the forces at a sphere of radius r, we see that, when $r = R$, the condition of continuity of the tangential components of the electric force gives the two equations

$$\left.\begin{aligned}\psi_n(\kappa R)\,\phi_n + E_n(\kappa R)\,\phi_n' &= \psi_n(\kappa' R)\,\phi_n'',\\ \left\{\frac{n}{2n+1}\psi_n(\kappa R)-\psi_{n-1}(\kappa R)\right\}\chi_n+\left\{\frac{n}{2n+1}E_n(\kappa R)-E_{n-1}(\kappa R)\right\}\chi_n' &\\ =\frac{1}{\mathrm{K}}\left\{\frac{n}{2n+1}\psi_n(\kappa' R)-\psi_{n-1}(\kappa' R)\right\}\chi_n'',&\end{aligned}\right\} \tag{22}$$

and the condition of continuity of the tangential components of the magnetic force gives two like equations obtained from the above by interchanging χ and ϕ, and writing $1/\mu$ for $1/\mathrm{K}$.

We observe that ϕ_n' and ϕ_n'' are determined in terms of ϕ_n, and χ_n' and χ_n'' in terms of χ_n. We eliminate ϕ_n'' and χ_n'' and obtain

$$\phi_n'\left[\frac{n}{2n+1}E_n(\kappa R)-E_{n-1}(\kappa R)+\frac{1}{\mu}\left\{\frac{\psi_{n-1}(\kappa' R)}{\psi_n(\kappa' R)}-\frac{n}{2n+1}\right\}E_n(\kappa R)\right]$$

$$=\phi_n\left[\psi_{n-1}(\kappa R)-\frac{n}{2n+1}\psi_n(\kappa R)-\frac{1}{\mu}\left\{\frac{\psi_{n-1}(\kappa' R)}{\psi_n(\kappa' R)}-\frac{n}{2n+1}\right\}\psi_n(\kappa R)\right], \tag{23}$$

and a like equation, having $1/\mathrm{K}$ in place of $1/\mu$, connects χ_n' with χ_n.

This completes the analytical solution of the problem.

5. To interpret the results we shall assume $\mu=1$ and investigate a first approximation and a second approximation when κR is small. In this case $\kappa' R$ also is small, and we may, for a first approximation, replace $\psi_n(\kappa R)$ and $\psi_n(\kappa' R)$ by unity for all values of n.

Introducing this value for the ψ functions into equation (23), and putting $\mu=1$, we see that ϕ_n' vanishes for all values of n. To express χ_n' we observe that, when κR is small, the approximate value of $E_n(\kappa R)$ is

$$(2n+1)\frac{\{1.3.5\ldots(2n-1)\}^2}{(\kappa R)^{2n+1}}e^{-\iota\kappa R}, \tag{24}$$

and we thus find, as the first approximation to χ_n',

$$\chi_n'=\frac{\dfrac{n+1}{2n+1}\left(1-\dfrac{1}{\mathrm{K}}\right)}{\dfrac{1}{2n+1}\left(n+\dfrac{n+1}{\mathrm{K}}\right)}\frac{(\kappa R)^{2n+1}e^{\iota\kappa R}\chi_n}{\{1.3.5\ldots(2n-1)\}^2(2n+1)}. \tag{25}$$

Now, referring to (14), and introducing the value of $P_n(z/r)$, we find

$$\left.\begin{aligned}V_0&=1, & V_2&=-\tfrac{1}{6}\kappa^2(2z^2-x^2-y^2),\\ V_1&=\iota\kappa z, & V_3&=-\frac{\iota\kappa^3}{30}\{2z^3-3z(x^2+y^2)\}.\end{aligned}\right\} \tag{26}$$

We have therefore, for a first approximation,

$$\phi_1=\frac{A\iota\kappa e^{\iota\kappa ct}}{2}x,\quad \chi_1=-\frac{A\iota\kappa e^{\iota\kappa ct}}{2}y,$$

and the most important terms in the expressions for the electric and

magnetic forces in the scattered wave are given by

$$\left.\begin{array}{l} X, Y, Z = \dfrac{-2}{\iota\kappa} E_0(\kappa r)\left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right)\chi_1' + \dfrac{1}{\iota\kappa}\dfrac{\kappa^2 r^5}{15} E_2(\kappa r)\left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right)\left(\dfrac{\chi_1'}{r^3}\right), \\ (\alpha, \beta, \gamma) = E_1(\kappa r)\left(y\dfrac{\partial}{\partial z} - z\dfrac{\partial}{\partial y},\ z\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial z},\ x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}\right)\chi_1'. \end{array}\right\} \tag{27}$$

To see what these become at a great distance from the sphere we observe that, when κr is great, the approximate value of $E_n(\kappa r)$ is

$$(\iota)^n 1.3 \ldots (2n+1)\frac{e^{-\iota\kappa r}}{(\kappa r)^{n+1}}, \tag{28}$$

and we hence find, as approximate forms for the electric and magnetic forces in the scattered wave,

$$\left.\begin{array}{l} (X, Y, Z) = \dfrac{K-1}{K+2}\dfrac{\kappa^2 R^3}{r} A e^{\iota\kappa(ct-r+R)}\left(-\dfrac{xy}{r^2}, \dfrac{x^2+z^2}{r^2}, -\dfrac{yz}{r^2}\right), \\ (\alpha, \beta, \gamma) = \dfrac{K-1}{K+2}\dfrac{\kappa^2 R^3}{r} A e^{\iota\kappa(ct-r+R)}\left(-\dfrac{z}{r}, 0, \dfrac{x}{r}\right). \end{array}\right\} \tag{29}$$

This agrees with Lord Rayleigh's approximate results, and shows that the disturbance in the scattered wave vanishes (to the order adopted) along the line $x = 0$, $z = 0$. It follows that, for very small particles, the scattered wave corresponding to an unpolarized train of incident waves should be polarized in a direction at right angles to the direction of propagation of the incident waves, and the plane of polarization should be parallel to this direction of propagation.

It is perhaps worthy of note that, to the lowest order the effect in the scattered wave at a distance, given by (29), is the same as that of a simple Hertzian oscillator with its axis parallel to the direction of the electric force in the incident waves.

6. For a second approximation the most important consideration is that ϕ_1' no longer vanishes. When terms of order $\kappa^2 R^2$ are retained

$$\psi_n(\kappa R) = 1 - \frac{\kappa^2 R^2}{2(2n+3)}, \tag{30}$$

and the coefficient of ϕ_n on the right-hand side of (23) becomes,

for $n=1$ and $\mu=1$,

$$1-\frac{\kappa^2R^2}{6}-\tfrac{1}{3}\left(1-\frac{\kappa^2R^2}{10}\right)-\left(1-\frac{\kappa^2R^2}{10}\right)\left\{\frac{1-\frac{\kappa^2R^2}{6}}{1-\frac{\kappa'^2R^2}{10}}-\tfrac{1}{3}\right\},$$

which is $-\frac{1}{15}(\kappa^2R^2-\kappa'^2R^2)$, or $\frac{K-1}{15}\kappa^2R^2$,

since $\kappa'^2=\kappa^2K$.

Hence, to order κ^5R^5, we have

$$\phi_1'=\frac{K-1}{45}\kappa^5R^5e^{\iota\kappa R}\phi_1, \tag{31}$$

and the additional terms thus introduced into X, Y, Z, α, β, γ at a great distance are

$$\left.\begin{aligned} &\text{for } (X, Y, Z) &&= -\frac{K-1}{30}\frac{\kappa^4R^5}{r}Ae^{\iota\kappa(ct-r+R)}\left(0,\ \frac{z}{r},\ -\frac{y}{r}\right),\\ &\text{and for } (\alpha, \beta, \gamma) &&= \frac{K-1}{30}\frac{\kappa^4R^5}{r}Ae^{\iota\kappa(ct-r+R)}\left(\frac{y^2+z^2}{r^2},\ -\frac{xy}{r^2},\ -\frac{xz}{r^2}\right).\end{aligned}\right\} \tag{32}$$

The introduction of these terms shows that the forces in the scattered wave vanish more nearly in a direction given by

$$x=0,\quad \frac{z}{r}=\frac{K+2}{30}\kappa^2R^2. \tag{33}$$

This is in accordance with the result of observation that for somewhat larger particles the scattered wave is more nearly polarized in a direction inclined at a slightly obtuse angle to the direction of propagation of the incident wave. It agrees also in general character, though not numerically, with Lord Rayleigh's second approximation.

7. The second approximation to the form of ϕ_1', which vanishes to a first approximation, has introduced into the expressions for the forces terms of the order κ^4R^5. It appears to be desirable to obtain expressions for the forces which shall be complete approximations of this order, that is shall contain all terms of order not exceeding κ^4R^5. To do this we shall require to carry the equation connecting χ_1' and χ_1 to a higher order than before, and we shall also need a second approximation to the value of χ_1; further, we shall have to investigate whether any of the higher harmonics $\phi_2 \ldots \chi_2 \ldots$ yield any terms of order κ^4R^5, and to evaluate any such terms if they can occur.

The complete expression for χ_1 is

$$\chi_1 = -\frac{A\iota\kappa e^{\iota\kappa ct}}{2\psi_1(\kappa R)}\left[\frac{R^2}{3}\psi_2(\kappa R)\frac{\kappa^2}{3}y + \psi_0(\kappa R)\,y\right]$$

$$= -\frac{A\iota\kappa e^{\iota\kappa ct}}{2}y\left(1+\frac{2}{45}\kappa^2R^2\right) \tag{34}$$

as far as terms of order κ^2R^2.

Again, we have, by (23),

$$\chi_1'\left[\tfrac{1}{3}E_1(\kappa R) - E_0(\kappa R) + \frac{1}{K}\left(\tfrac{2}{3} - \frac{\kappa'^2R^2}{15}\right)E_1(\kappa R)\right]$$

$$= \chi_1\left[1 - \frac{\kappa^2R^2}{6} - \tfrac{1}{3}\left(1-\frac{\kappa^2R^2}{10}\right) - \frac{1}{K}\left(\tfrac{2}{3} - \frac{\kappa'^2R^2}{15}\right)\left(1-\frac{\kappa^2R^2}{10}\right)\right], \tag{35}$$

where we have put in the second approximation to the ψ functions, and we have to put

$$E_0(\kappa R) = \frac{e^{-\iota\kappa R}}{\kappa R}, \quad E_1(\kappa R) = 3\,\frac{1+\iota\kappa R}{\kappa^3R^3}e^{-\iota\kappa R}. \tag{36}$$

We find, to the order κ^5R^5,

$$\chi_1' = \frac{K-1}{K+2}\tfrac{2}{3}\kappa^3R^3e^{\iota\kappa R}\left[1 - \iota\kappa R - \kappa^2R^2\left\{\tfrac{11}{10} - \frac{6K}{5(K+2)}\right\}\right]\chi_1. \tag{37}$$

Hence so far as χ_1 is concerned the expressions obtained in the first approximations to the forces are to be multiplied by

$$1 - \iota\kappa R - \kappa^2R^2\left\{\tfrac{11}{10} - \frac{6K}{5(K+2)}\right\} + \tfrac{2}{45}\kappa^2R^2.$$

Again, we find that the terms of lowest order in χ_2' are given by the approximate forms

$$\left.\begin{aligned}\chi_2 &= \frac{A\kappa^2e^{\iota\kappa ct}}{6}yz,\\ \chi_2' &= \frac{3(K-1)}{2K+3}\,\frac{\kappa^5R^5e^{\iota\kappa R}}{45}\chi_2,\end{aligned}\right\} \tag{38}$$

and we have contributions to the expressions for X, Y, Z and (α, β, γ) in the scattered wave of the forms

$$\left.\begin{aligned}(X, Y, Z) &= \frac{-3}{\iota\kappa}E_1(\kappa r)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\chi_2' + \frac{2}{\iota\kappa}\,\frac{\kappa^2r^7}{35}E_3(\kappa r)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(\frac{\chi_2'}{r^5}\right),\\ (\alpha, \beta, \gamma) &= E_2(\kappa r)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},\; z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},\; x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\chi_2'.\end{aligned}\right\} \tag{39}$$

The terms thus contributed to the expressions for (X, Y, Z) at a great distance are

$$\frac{K-1}{2K+3}\,\frac{\kappa^4 R^5}{r}\,Ae^{\iota\kappa(ct-r+R)}\left[\tfrac{1}{3}\frac{yz}{r^2}\left(\frac{x}{r},\ \frac{y}{r},\ \frac{z}{r}\right)-\tfrac{1}{6}\left(0,\ \frac{z}{r},\ \frac{y}{r}\right)\right]. \quad (40)$$

In like manner the terms thus contributed to the expressions for (α, β, γ) at a great distance are

$$\frac{K-1}{2K+3}\,\frac{\kappa^4 R^5}{r}\,Ae^{\iota\kappa(ct-r+R)}\,\tfrac{1}{6}\left(0,\ \frac{xy}{r^2},\ \frac{-xz}{r^2}\right). \quad (41)$$

It appears on inspection of the ratios ϕ_2'/φ_2 and χ_3'/χ_3 that we have now exhausted all the terms of order $\kappa^4 R^5$.

Thus the complete expressions, as far as terms of order $\kappa^4 R^5$, for the forces in the scattered wave at a great distance are

$$\begin{aligned}(X, Y, Z) = {} & \frac{K-1}{K+2}Ae^{\iota\kappa(ct-r+R)}\left(-\frac{xy}{r^2},\ \frac{x^2+z^2}{r^2},\ -\frac{yz}{r^2}\right)\frac{\kappa^2 R^3}{r} \\ & \times\left\{1-\iota\kappa R-\kappa^2R^2\left(\tfrac{19}{18}-\tfrac{6}{5}\frac{K}{K+2}\right)\right\} \\ & +(K-1)\,Ae^{\iota\kappa(ct-r+R)}\frac{\kappa^4R^5}{r}\tfrac{1}{30}\left(0,\ -\frac{z}{r},\ \frac{y}{r}\right) \\ & +\frac{K-1}{2K+3}\,Ae^{\iota\kappa(ct-r+R)}\frac{\kappa^4R^5}{r}\,\tfrac{1}{6}\left(0,\ -\frac{z}{r},\ -\frac{y}{r}\right) \\ & +\frac{K-1}{2K+3}\,Ae^{\iota\kappa(ct-r+R)}\frac{\kappa^4R^5}{r}\tfrac{1}{3}\frac{yz}{r^2}\left(\frac{x}{r},\ \frac{y}{r},\ \frac{z}{r}\right), \end{aligned} \quad (42)$$

and

$$\begin{aligned}(\alpha, \beta, \gamma) = {} & \frac{K-1}{K+2}Ae^{\iota\kappa(ct-r+R)}\left(-\frac{z}{r},\ 0,\ \frac{x}{r}\right)\frac{\kappa^2 R^3}{r} \\ & \times\left\{1-\iota\kappa R-\kappa^2R^2\left(\tfrac{19}{18}-\tfrac{6}{5}\frac{K}{K+2}\right)\right\} \\ & +(K-1)\,Ae^{\iota\kappa(ct-r+R)}\frac{\kappa^4R^5}{r}\tfrac{1}{30}\left(\frac{y^2+z^2}{r^2},\ -\frac{xy}{r^2},\ -\frac{xz}{r^2}\right) \\ & +\frac{K-1}{2K+3}\,Ae^{\iota\kappa(ct-r+R)}\frac{\kappa^4R^5}{r}\tfrac{1}{6}\left(0,\ \frac{xy}{r^2},\ -\frac{xz}{r^2}\right). \end{aligned} \quad (43)$$

The results are in agreement with observation inasmuch as they show that there is no direction in which the forces in the scattered wave completely vanish, and that the intensity of the residual disturbance in the direction in which it most nearly vanishes varies *inversely* as the eighth power of the wave-length. It is noteworthy

that, if terms of order $\kappa^3 R^4$ are retained and terms of order $\kappa^4 R^5$ neglected, the scattered wave is precisely as given by the first approximation (29), except that the exponential factor becomes $e^{\iota\kappa(ct-r)}$.

8. A very similar analysis applies to the problem when the material of the sphere is regarded as conducting. If σ is the specific resistance, the equations that hold within the sphere become

$$\left.\begin{aligned}\left(\frac{\mathrm{K}}{c}\frac{\partial}{\partial t}+\frac{4\pi c}{\sigma}\right)(X,\ Y,\ Z) &= \operatorname{curl}(\alpha,\ \beta,\ \gamma),\\ -\frac{\mu}{c}\frac{\partial}{\partial t}(\alpha,\ \beta,\ \gamma) &= \operatorname{curl}(X,\ Y,\ Z),\end{aligned}\right\} \tag{44}$$

and it follows that, when all the forces are proportional to $e^{\iota\kappa ct}$, each of them satisfies an equation of the form

$$(\nabla^2+\kappa'^2)\,u = 0,$$

where
$$\kappa'^2 = \kappa^2\mathrm{K}\mu - \frac{4\pi\mu\,\iota\kappa c}{\sigma} = \kappa^2\mu\left\{\mathrm{K}-\frac{4\pi\iota c}{\kappa\sigma}\right\}. \tag{45}$$

The two circuital relations hold as before, and the forces at points within the sphere are connected by the equations

$$\left.\begin{aligned}(X,\ Y,\ Z) &= \frac{1}{\iota\kappa\mathrm{K}+4\pi c/\sigma}\operatorname{curl}(\alpha,\ \beta,\ \gamma),\\ (\alpha,\ \beta,\ \gamma) &= \frac{-1}{\iota\kappa\mu}\operatorname{curl}(X,\ Y,\ Z).\end{aligned}\right\} \tag{46}$$

The forms of the expressions for the forces at internal points are easily written down in terms of two systems of solid harmonics ϕ_n'' and χ_n'', and the forms at external points are the same as before. In the boundary conditions we have to put

$$\frac{1}{\mathrm{K}-4\pi\iota c/\kappa\sigma} \text{ in place of } \frac{1}{\mathrm{K}},$$

wherever $1/\mathrm{K}$ occurs explicitly.

The result of the first approximation still holds good, provided $|\kappa' R|$ is small when κR is small. With frequencies of orders of magnitude corresponding to visible light K is the square of the refractive index, and the quantity $4\pi c/\kappa\sigma$ is small compared with K for badly conducting materials, and thus a slight degree of

conductivity does not appreciably affect the result.* For a good conductor, however, K may be omitted, and

$$\kappa'^2 = -\frac{4\pi\mu\iota\kappa c}{\sigma}.$$

I find that for frequency and wave-length corresponding to the D lines of the solar spectrum, and with σ equal to the specific resistance of copper,

$$\frac{\kappa'}{\kappa} = \surd(-\iota)\, 45 \text{ nearly.}$$

To make $|\kappa' R|$ small we should require $45\kappa R$ to be small, or, since $\kappa = 2\pi/\lambda$, where λ is the wave-length, this would require R to be about $\lambda/300$ to make $|\kappa' R|$ about 1/10. Such a value of R is so near to molecular dimensions that a continuous analysis could not be applied to the problem. On the other hand, if we could imagine the resistance to be very much less than it is for the best conductors, we might make an approximation on the supposition that $\kappa' R$ is great while κR is small.

In writing out this approximation we put $\mu = 1$, and

$$\kappa' = \frac{1-\iota}{\surd 2}\sqrt{\left(\frac{4\pi c}{\kappa\sigma}\right)} = (1-\iota)\,\vartheta \text{ say;}$$

then
$$\psi_0(\kappa' R) = \frac{\sin \kappa' R}{\kappa' R} = \frac{e^{\vartheta R} e^{\iota\vartheta R}}{2\iota\kappa' R}$$

approximately when ϑR is great. Also we have

$$\psi_1(\kappa' R) = -\frac{3}{\kappa'^3 R^3}(\kappa' R \cos\kappa' R - \sin\kappa' R)$$

$$= -\frac{3}{\kappa'^3 R^3}\,\tfrac{1}{2}\,(\kappa' R + \iota)\, e^{\vartheta R} e^{\iota\vartheta R}$$

approximately when ϑR is great. This gives

$$\frac{\psi_0(\kappa' R)}{\psi_1(\kappa' R)} = \frac{\kappa'^2 R^2}{3}\,\frac{1}{1-\iota\kappa' R} = \frac{\iota}{3}\kappa' R \tag{47}$$

approximately when $\kappa' R$ is great.

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* *Cf.* G. W. Walker, *Quarterly Journal of Pure and Applied Mathematics*, Vol. xxx., 1898, p. 217.

The equation connecting χ_1' and χ_1 now becomes

$$\chi_1'\left[\tfrac{1}{3}E_1(\kappa R)-E_0(\kappa R)+\frac{\kappa^2}{\kappa'^2}\left(\frac{\iota}{3}\kappa' R\right)E_1(\kappa R)\right]=\chi_1\left[\tfrac{2}{3}-\frac{\kappa^2}{\kappa'^2}\left(\frac{\iota}{3}\kappa' R\right)\right],$$

where we are to put

$$E_0(\kappa R)=\frac{e^{-\iota\kappa R}}{\kappa R},\quad E_1(\kappa R)=\frac{3}{\kappa^3R^3}e^{-\iota\kappa R}.$$

Thus the most important part of χ_1' is given by

$$\chi_1'=\tfrac{2}{3}\kappa^3R^3e^{\iota\kappa R}\chi_1. \tag{48}$$

Again, the equation connecting ϕ_1' and ϕ_1 becomes

$$\phi_1'=-\tfrac{1}{3}\kappa^3R^3e^{\iota\kappa R}\phi_1, \tag{49}$$

when only the most important terms are retained.

With these approximations the expressions for the forces at a distance become

$$\begin{aligned}(X,Y,Z)=&\frac{\kappa^2R^3}{r}Ae^{\iota\kappa(ct-r+R)}\left(-\frac{xy}{r^2},\ \frac{x^2+z^2}{r^2},\ -\frac{yz}{r^2}\right)\\&+\tfrac{1}{2}\frac{\kappa^2R^3}{r}Ae^{\iota\kappa(ct-r+R)}\left(0,\ \frac{z}{r},\ -\frac{y}{r}\right),\end{aligned} \tag{50}$$

and

$$\begin{aligned}(\alpha,\beta,\gamma)=&\frac{\kappa^2R^3}{r}Ae^{\iota\kappa(ct-r+R)}\left(-\frac{z}{r},\ 0,\ \frac{x}{r}\right)\\&-\tfrac{1}{2}\frac{\kappa^2R^3}{r}Ae^{\iota\kappa(ct-r+R)}\left(\frac{y^2+z^2}{r^2},\ -\frac{xy}{r^2},\ -\frac{xz}{r^2}\right).\end{aligned} \tag{51}$$

These expressions verify Prof. J. J. Thomson's result* that, for a perfect conductor, the forces in the scattered wave vanish in the direction $x=0$, $z/r=-\frac{1}{2}$, *i.e.*, in a direction making an angle $\frac{1}{3}\pi$ with the direction of propagation of the incident waves. This result could have been more simply obtained by neglecting the disturbance inside the sphere and taking the electric force at the boundary to be purely radial. We have shown above that there is no reason for thinking that this investigation could have any application to the problem of the scattering of light by small particles, though it might conceivably represent something that could be observed for Hertzian waves a metre long and metallic spheres of a few millimetres radius.

* *Recent Researches*, p. 448.

The Jacobian Locus in Hyper-geometry. *By* P. H. SCHOUTE.
Received February 8th, 1899. Read February 9th, 1899. Received, in revised form, March 20th, 1899.

1. We represent by a_c^b any geometric manifold a of b dimensions and order c, possible in our field of operation, a linear hyper-space of n dimensions, and by $(a_c^b)_d$ a linear system of these manifolds with d degrees of freedom. For the general part a common to these symbols, we substitute l in the case of linear manifolds, as lines, planes, linear spaces, and hyper-spaces, q in the case of quadratic manifolds as conics, quadrics, and hyper-quadrics, and c in the particular case of manifolds with a double point. Moreover, we use a capital A, L, Q, C, when the corresponding manifold is specially indicated. So L_1^n is the symbol for our field of operation, and q_2^{n-1}, c_2^{n-1}, C_3^{n-1} stand respectively for any hyper-quadric, any hyper-cone, a determinate hyper-cubic with a double point, all of $n-1$ dimensions, contained in it.

The linear polars l_1^{n-1} of any point P with regard to the hyper-spaces of a given linear system $(a_m^{n-1})_\mu$ intersect in a linear manifold $l^{n-(\mu+1)}$. The locus of the point P, whose linear polars with regard to these hyper-spaces have an $l^{n-\mu}$ in common—*i.e.*, the locus of the double points, or the Jacobian locus, of the linear system—is a hyper-space of $n-1$ dimensions, whose order can be deduced from a very general* and, at the same time, most simple theorem given by Salmon (see the nineteenth lesson of his *Modern Higher Algebra*, fourth edition, Art. 272), which runs as follows:—

If the $r\rho$ elements of the matrix

$$\left\|\begin{matrix} a_1+\alpha_1 & a_2+\alpha_1 & \dots & a_e+\alpha_1 & \dots & a_r+\alpha_1 \\ a_1+\alpha_2 & a_2+\alpha_2 & \dots & a_e+\alpha_2 & \dots & a_r+\alpha_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_1+\alpha_\epsilon & a_2+\alpha_\epsilon & \dots & a_e+\alpha_\epsilon & \dots & a_r+\alpha_\epsilon \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_1+\alpha_\rho & a_2+\alpha_\rho & \dots & a_e+\alpha_\rho & \dots & a_r+\alpha_\rho \end{matrix}\right\|$$

* As is easily recognized, the order arrangement given in the matrix, though not the most general *per se*, is the most general one for which all the determinants of order r contained in the matrix of the forms themselves are all homogeneous in the $n+1$ variables $x_1, x_2, \dots, x_{n+1}$.

with r columns and ρ rows ($r \geqq \rho$) indicate the orders of $r\rho$ given homogeneous forms $f_{e,\epsilon}$ in $n+1$ variables $x_1, x_2, \ldots x_{n+1}$, then the order of the system obtained by putting equal to nought the $r-\rho+1$ mutually independent determinants of order ρ contained in the corresponding matrix of the forms $f_{e,\epsilon}$ themselves is represented by

$$C_{r-\rho+1}+C_{r-\rho}H_1+C_{r-\rho-1}H_2+\ldots+H_{r-\rho+1},$$

where $\qquad C_1 = \Sigma a_1, \quad C_2 = \Sigma a_1 a_2, \quad C_3 = \Sigma a_1 a_2 a_3, \; \ldots,$

and $\quad H_1 = \Sigma \alpha_1, \quad H_2 = \Sigma \alpha_1^2 + \Sigma \alpha_1 \alpha_2, \quad H_3 = \Sigma \alpha_1^3 + \Sigma \alpha_1^2 \alpha_2 + \Sigma \alpha_1 \alpha_2 \alpha_3, \; \ldots .$

In the present case we evidently have to deal with the $(n+1)(\mu+1)$ first differential quotients with respect to the $n+1$ variables $x_1, x_2, \ldots, x_{n+1}$ of $\mu+1$ forms A_m^{n-1}, determining the linear system $(a_m^{n-1})_\mu$. As this system only gives rise to a Jacobian locus under the condition $n \geqq \mu$, we are obliged, in conformity with the notation of the matrix, to place the $n+1$ differential quotients of the same form A_m^{n-1} in the same row. So we have

$$r = n+1 \quad \text{and} \quad \rho = \mu+1.$$

Then it is convenient to put

$$a_1 = a_2 = \ldots = a_{n+1} = m-1 \quad \text{and} \quad \alpha_1 = \alpha_2 = \ldots = \alpha_{\mu+1} = 0,$$

which leads to the result $C_{n-\mu+1}$, that easily proves to be equal to $(m-1)^{n-\mu+1}(n+1)_\mu$, where, as usual, $(n+1)_\mu$ indicates the μ^{th} binomial coefficient of $(a+b)^{n+1}$, the coefficient unity of a^{n+1} being considered as $(n+1)_0$.

2. The aim of this paper is to consider the case $m=2$ of a linear system $(q_2^{n-1})_\mu$ of hyper-quadrics geometrically, by reasoning in the space L_1 itself. Thereby we shall find another way to the result $(n+1)_\mu$ for the order of the Jacobian locus of the linear system $(q_2^{n-1})_\mu$, *i.e.* of the vertices of the hyper-cones contained in it. We discovered it before our attention was kindly directed to Salmon's most beautiful theorem.* In publishing it here we only intend to serve didactic purposes with regard to hyper-geometric discussion; moreover, the hope is not excluded that it may be hereafter extended to the more general case of the linear system $(a_m^{n-1})_\mu$.

The right line L_1^1 common to the polar planes of the vertex V of a

* We have to thank the referee, Mr. A. Berry, for this reference.

cone belonging to the net $(q_2^2)_2$ in L_1^3 with regard to the quadrics of this net, is a trisecant of the Jacobian skew sextic of the net; so any plane l_1^2 through this line L_1^1 cuts the Jacobian curve in two sets of three points, one set on L_1^1, one set elsewhere. These generally known facts form the germs of the geometric method in view, for the relation $6 = 3+3$, on which it depends, proves to be a particular case of the general law of recurrency

$$(n+1)_\mu = (n)_{\mu-1} + (n)_\mu,$$

which connects binomial coefficients. In other words, if without any reference to these coefficients we represent by (n, μ) the order of the Jacobian locus of the linear system $(q_2^{n-1})_\mu$ in L_1^n, then we find equally

$$(n, \mu) = (n-1, \mu-1) + (n-1, \mu).$$

We now proceed to the demonstration of this law.

If, for shortness sake, the linear hyper-space common to the polars l_1^{n-1} of any point P with regard to all the hyper-spaces q_2^{n-1} of the linear system $(q_2^{n-1})_\mu$ be called the "polar figure" of $(q_2^{n-1})_\mu$ for P, it is easily shown that this polar figure is an $l_1^{n-\mu-1}$ or an $L_1^{n-\mu}$, according as P is a point chosen at random or the vertex of any hyper-cone c_2^{n-1} belonging to the system. In the first case the polar figure is the intersection of $\mu+1$ hyper-spaces l_1^{n-1} independent of one another, the polars of P with regard to any set of $\mu+1$ hyper-quadrics q_2^{n-1} of the given $(q_2^{n-1})_\mu$ not belonging to a $(q_2^{n-1})_{\mu-1}$ contained in it. But, in the case of a double point C of the system, the polar of C with respect to the c_2^{n-1} of which it is the vertex is indeterminate. By specializing the set of $\mu+1$ hyper-quadrics, mentioned just now, so as to include this c_2^{n-1}, it is seen that the polar figure of C is the intersection of only μ hyper-spaces l_1^{n-1}, independent of one another, and therefore an $L_1^{n-\mu}$. At any rate, if the polar figure is given, there is only one pole P or C to which it belongs, for the polar figure is determined by $n-\mu$ or more mutually independent points, and the polars l_1^{n-1} of these points with regard to the hyper-quadrics of the given system have only one point in common.

Any determinate hyper-space $L_1^{n-\mu+1}$ through the polar figure $L_1^{n-\mu}$ of the double point C of the system cuts the Jacobian locus under discussion in a certain number of points J, the sum of the numbers $\mu-1$ and $n-\mu+1$ of the dimensions of this locus and of $L_1^{n-\mu+1}$ being n; *according* to definition, this number of points, indicating the order

of the Jacobian locus, may be represented by the symbol (n, μ). These (n, μ) points J will be shown to break up into two different groups, a group of $(n-1, \mu-1)$ points J' in $L_1^{n-\mu+1}$ not situated in $L_1^{n-\mu}$, and a group of $(n-1, \mu)$ points J'' in $L_1^{n-\mu}$; this will complete the proof of the general law of recurrency.

But before we proceed to this we have to seek for a useful discriminating characteristic of quadratic hyper-cones in L_1^n. In ordinary space, a quadric through V must be a cone with vertex V, if it touches in V a plane and a line not contained in that plane. In the same manner a hyper-quadric q_2^{n-1} in L_1^n through V must be a hyper-cone with vertex V, if it touches in V a linear hyper-space L_1^{n-1} and a line not contained in this hyper-space. This remark induces us to pass through $L_1^{n-\mu+1}$ any determinate linear hyper-space L_1^{n-1}, and to consider the hyper-cones c_2^{n-2} of the intersection $(q_2^{n-2})_\mu$ of the given system with this L_1^{n-1}.

The number of the points J' is $(n-1, \mu-1)$.—The linear hyper-space $L_1^{n-\mu+1}$ through the polar figure $L_1^{n-\mu}$ of the double point C of the system is itself the polar figure of this point C with regard to the hyper-quadrics of a linear system $(q_2^{n-1})_{\mu-1}$ contained in the given system. To prove this, let D be a determinate point of $L_1^{n-\mu+1}$ and $P_1, P_2, \ldots, P_{\mu-1}$ points chosen at random in L_1^{n-1}. Then the hyper-quadrics of the given system that pass through $P_1, P_2, \ldots, P_{\mu-1}$ form a pencil $(q_2^{n-1})_1$ and this pencil contains always one, and only one, hyper-quadric q_2^{n-1} for which C and D are conjugate to each other, and the polar of C passes through D; in other words, of the hyper-quadrics of $(q_2^{n-1})_\mu$, for which C and $L_1^{n-\mu+1}$ are pole and polar, one, and only one, passes through $\mu-1$ points P chosen at random in L_1^n *i.e.*, the hyper-quadrics of the given system for which C and $L_1^{n-\mu+1}$ are pole and polar form a linear system $(q_2^{n-1})_{\mu-1}$ with $\mu-1$ degrees of freedom. Now the linear hyper-space L_1^{n-1} through $L_1^{n-\mu+1}$ cuts the linear system $(q_2^{n-1})_{\mu-1}$, obtained just now, in a linear system $(q_2^{n-2})_{\mu-1}$, and any C_2^{n-2} of this, that is a hyper-cone whose vertex J' is a point of $L_1^{n-\mu+1}$, can be shown to be the intersection of L_1^{n-1} with a hyper-quadric of the linear system $(q_2^{n-1})_{\mu-1}$ possessing likewise a double point in J'. In fact, this q_2^{n-1} is touched in J' by L_1^{n-1}, and by the line CJ' not contained in that hyper-space of $n-1$ dimensions. For L_1^{n-1} cuts it in a C_2^{n-2} with a double point in J', and J' is a point

of the polar $L_1^{n-\mu+1}$ of C with respect to it. So we find that the number of the points J' contained in $L_1^{n-\mu+1}$ is equal to that of the points of intersection of $L_1^{n-\mu+1}$ with the Jacobian locus of a linear system $(q_2^{n-2})_{\mu-1}$ in L_1^{n-1}, *i.e.*, to the order of this locus, and therefore to $(n-1, \mu-1)$, according to definition. And the mobility of $L_1^{n-\mu+1}$ about $L_1^{n-\mu}$ proves that in general none of these $(n-1, \mu-1)$ points J' belongs to $L_1^{n-\mu}$.

The number of the points J'' *is* $(n-1, \mu)$.—The linear hyper-space L_1^{n-1} through $L_1^{n-\mu+1}$ through $L_1^{n-\mu}$ cuts the given system $(q_2^{n-1})_\mu$ itself in a linear system $(q_2^{n-2})_\mu$. Any hyper-cone C_2^{n-2} of the last system having a point J'' of $L^{n-\mu}$ for vertex is the intersection of L_1^{n-1} with a hyper-quadric of the given system possessing likewise a double point in J''. Indeed, this hyper-quadric is touched in J'' by L_1^{n-1}, and by the line CJ'' not contained in L_1^{n-1}. For L_1^{n-1} cuts it in a hyper-cone C_2^{n-2} with vertex J'', and J'' is a point of the polar $L_1^{n-\mu}$ of C with respect to it. Therefore the number of the points J'' contained in $L_1^{n-\mu}$ is equal to that of the points common to $L_1^{n-\mu}$, and the Jacobian locus of the linear system $(q_2^{n-})_\mu$ in L_1^{n-1}, *i.e.*, to the order of this locus, *i.e.*, to $(n-1, \mu)$.

Equivalence of (n, μ) *and* $(n+1)_\mu$.—What we have now proved geometrically is this. If it be known that with regard to the system $(q_2^{n-2})_\mu$ in L_1^{n-1} the orders of the different Jacobian loci corresponding to the values 1, 2, ..., $n-1$ of μ are given by the series $(n)_1$, $(n)_2$, ..., $(n)_{n-1}$ of binomial coefficients, then the orders of the different Jacobian loci of the system $(q_2^{n-1})_\mu$ in L_1^n for the values 2, 3, ..., $n-1$ of μ will be represented by the series $(n+1)_2$, $(n+1)_3$, ..., $(n+1)_{n-1}$. Now, in order to be able to complete the induction from n to $n+1$, we have to seek the bordering terms $(n, 1)$ and (n, n) that correspond to the values 1 and n of μ. In the case of the pencil $(q_2^{n-1})_1$ the double points of the system are at the same time the $n+1$ vertices of the common self-polar *simplicissimum* with regard to the hyper-quadrics of the pencil; as is generally known, this *simplicissimum* of L_1^n corresponds to the triangle in the plane, the tetrahedron in ordinary space, the pentahedroid in L_1^4, &c. Or, more in accordance with the indicated geometric method, if L_1^{n-1} is the polar figure of a double point C of the pencil, then the order of the locus—*i.e.*, here the number of the double points—is found as the sum of one point J' in L^n *through* L_1^{n-1} (viz., the point C itself), and n points J'' in L_1^{n-1}.

And in the case of the system $(q_2^{n-1})_n$ the locus required is the locus of the point common to $n+1$ corresponding elements l_1^{n-1} of $n+1$ projective linear systems $(l_1^{n-1})_n$, the linear systems formed by the polars of $n+1$ mutually independent points $P_1, P_2, \ldots, P_{n+1}$ of L_1^n with regard to the given linear system $(q_2^{n-1})_n$.

Completed in this manner, the conclusion from n to $n+1$ forms a rigorous demonstration, for the final result $(n+1)_\mu$ holds for $n = 1$, &c.

3. At first it would seem that the same manner of reasoning could be applied to the case of hyper-spaces a_m^{n-1} by very slight alterations, as the substitution of hyper-space C_m^{n-1} with a double point, and linear polar for hyper-cone C_2^{n-1}, and polar, &c. For in this more general case we can also reduce the determination of the order of the Jacobian locus to that of two numbers of points, a group of points J'' in $L_1^{n-\mu}$, polar figure of any double point C of the system, and a group of points J' in any hyper-space $L_1^{n-\mu+1}$ through $L_1^{n-\mu}$. Indeed, it might seem that the two numbers alluded to are quite independent of m, which supposition would lead to the same law of recurrency. But this is evidently wrong. For according to the general result of art. 1, the law of recurrency, we ought to find, is different, viz.,

$$(n, \mu) = (n-1, \mu-1)+(m-1)(n-1, \mu).$$

A closer inspection soon shows us at least one way out of this difficulty. It lies in the fact that any polar figure $L_1^{n-\mu-1}$ with regard to the linear system $(a_m^{n-1})_\mu$ belongs in general to several poles P, unless $m = 2$, which remark gives room to the hypothesis that the polar figure $L_1^{n-\mu}$ of any double point C of the system may belong to different double points. And in that case a determinate $L_1^{n-\mu+1}$ through $L_1^{n-\mu}$ corresponds to several linear systems $(a_m^{n-2})_{\mu-1}$ contained in the given system, &c.

But this argument, though it has strengthened our conviction that the geometric treatment of the case $m = 2$ is a rigorous one, has not brought us one step further in the direction of the extension required, which, according to our opinion, is a desideratum for two reasons. First and chiefly, as without doubt it will reveal interesting properties forming the generalization of this, that the Jacobian locus corresponding to $(q_2^{n-1})_\mu$ admits of a $\mu-1$-fold infinite system of linear hyper-spaces $L_1^{n-\mu}$ cutting it in sets of $(n)_{\mu-1}$ points, the polar figures

of its points J, &c. Moreover, in the second place, the remarkably general theorem of Salmon, which in itself is a perfect piece of work, has a drawback in its demonstration, consisting as it does of a verification of a number of particular cases followed by an appeal to analogy; so that it would at least be very convenient to possess a more rigorous demonstration of the result $(m-1)^{n-\mu+1}(n+1)_\mu$ quite independent of that theorem. It is in the hope that some reader of the present note may be able to supply this desideratum that we have thought it desirable to exhibit in this third article our embryonic views about the extension.

Thursday, March 9th, 1899.

Lt.-Col. A. J. C. CUNNINGHAM, R.E., Vice-President, in the Chair.

Fifteen members present.

The following gentlemen were elected members:—Prof. Leonard Eugene Dickson, University of California, Berkeley, U.S.A.; Prof. Alfred Cardew Dixon, Sc.D., Queen's College, Galway; and Mr. Harold Hilton, B.A., Fellow of Magdalen College, Oxford.

Mr. J. G. Leathem was admitted into the Society.

Dr. Larmor made a few remarks "On the Phenomenon of Zeeman and its bearing on the Problem of the Origin of Spectra." Messrs. Hargreaves and Hobson spoke on the subject of the communication.

Dr. Macaulay gave an account of a "Note on Involution," by Mr. G. B. Mathews.

The following papers were given in abstract:—

Note on the Expansion of $\tan(\sin\theta)-\sin(\tan\theta)$ in powers of θ: Mr. R. H. Pinkerton.

Note on a Property of Groups of Prime Degree: Prof. W. Burnside.

Note on the Invariant Total Differential Equation in Three Variables: Prof. J. M. Page.

The following presents were made to the Library :—

"Mathematical Magazine," Vol. II., No. 11; Washington, 1898.

Hankel, W. G.—"Elektrische Untersuchungen," roy. 8vo; Leipzig, 1899.

"Wiadomosci Matematyczne," Tom I., Zeszyt 1–5, 1897; Tom II., Zeszyt 1–6, 1898; Warsaw.

Sommerfeld, A.—"Ueber die Fortpflanzung elektrodynamischer Wellen längs eines Drahtes," 8vo; Leipzig, 1899.

"Educational Times," March, 1899.

"Indian Engineering," Vol. XXV., Nos. 3–6; Jan. 21–Feb. 11, 1899.

The following exchanges were received :—

"Proceedings of the Royal Society," Vol. LXIV., Nos. 407–8, 1899.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXIII., St. 2; Leipzig, 1899.

"Bulletin of the American Mathematical Society," Series 2, Vol. V., No. 5, February, 1899; New York.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. XIII., No. 5; Coimbra, 1899.

"Bulletin des Sciences Mathématiques," Tome XXIII., January, 1899; Paris.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. V., Fasc. 1, January, 1899; Napoli.

"Archives Néerlandaises," Série 2, Tome II., Livr. 2–4; La Haye, 1899.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. VIII., Fasc. 2, 3; Roma, 1899.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig," Band L., 1898.

"Nyt Tidsskrift for Matematik," A, Aarg. 10, Nr. 1, 2; B, Aarg. 10, Nr. 1; Copenhagen, 1899.

"Nieuw Archief voor Wiskunde," 2 Reeks, Deel IV., St. 1; Amsterdam, 1899.

"Sitzungsberichte der Königl. Preuss. Akademie der Wissenschaften zu Berlin," XL.–LIV.; 1898.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," Heft 4; 1898.

APPENDIX.

The following resolution of Council was passed on January 12th, 1899 :—

> "That in future the volumes of *Proceedings* shall contain as nearly 400 pages as may be found convenient, provided that each volume shall begin with the Report of Proceedings at a meeting, not necessarily an Annual General Meeting."

The following "Explanatory Note and Correction" has been received from Mr. Hugh MacColl :*—

On page 102 of my Seventh Paper (*Proc. Lond. Math. Soc.*, Vol. XXIX.) I said that $A^x : B^y$ was synonymous with $\left(\frac{B}{A^x} = y\right)$ and with $\left(\frac{B^y}{A^x} = 1\right)$. This is an error; the third expression always implies both the first and second, but it is not formally equivalent to either. To prove this, let ϕ_1, ϕ_2, ϕ_3 respectively denote the three statements asserted to be synonyms. Also let α denote A^x, and let β denote B^y. By definition we have

$$\phi_1 = \alpha : \beta = (\alpha\beta')^\eta = \left(\frac{\alpha\beta'}{\epsilon} = 0\right) = \left(\frac{\alpha}{\epsilon}\,\frac{\beta'}{\alpha} = 0\right)$$

$$= \left\{\frac{\alpha}{\epsilon}\left(1 - \frac{\beta}{\alpha}\right) = 0\right\} = \left(\frac{\alpha}{\epsilon} = 0\right) + \left(1 - \frac{\beta}{\alpha} = 0\right)$$

$$= \alpha^\eta + \left(\frac{\beta}{\alpha} = 1\right) = A^{x\eta} + \phi_3.$$

Thus we see that ϕ_3 always implies ϕ_1, but is not equivalent to ϕ_1, except when A and x are of such a nature that we have

$$A^{x\eta} = \eta.$$

The statements ϕ_2 and ϕ_3 appear to have the same relations to each other as A and A^ϵ. For, assuming $A^x = \epsilon$, we get

$$\phi_2 = \left(\frac{B}{A^x} = y\right) = \left(\frac{B}{\epsilon} = y\right) = B^y, \text{ by definition;}$$

* Received April 12th, 1899.

whereas

$$\phi_3 = \left(\frac{B^y}{A^x} = 1\right) = \left(\frac{B^y}{\epsilon} = 1\right) = B^{y\epsilon}, \text{ by definition.}$$

Thus, in asserting ϕ_2 to be synonymous with ϕ_3, I have fallen into an error like that which, in my Fifth Paper (*Proc. Lond. Math. Soc.*, Vol. XXVIII., p. 182), I pointed out in the criticisms of Drs. Venn and Schröder, namely, that of confounding the statement a with the statement a^ϵ.

To show how a statement of any degree, such as A^{abcd}, may arise in probability, let us suppose A to be taken at random out of a series A_1, A_2, A_3, &c. If *every* statement of the series be true, we have A^ϵ. If the series contain in all n statements out of which an are true and $(1-a)\,n$ false, we have A^a, which (when $a > 0$ and < 1) implies A^θ. If all the statements are false, we have A^η. Thus A^a, whether a denotes ϵ or η or θ, is a conclusion founded upon our knowledge of A_1, A_2, A_3, &c. But different data might lead to the same conclusion or to a different conclusion. Let the data K_1 lead to the conclusion A^a; K_2 to the conclusion $A^{a'}$ (the denial of A^a); and K_3 to the conclusion A^a. Let K be taken at random out of the three collections K_1, K_2, K_3. Then we get A^{ab}, in which $b = \frac{2}{3}$. But, if K_2 had led to the conclusion A^a, we should have had $b = 1$, and therefore $A^{a\epsilon}$. By continuing this reasoning we can interpret the meaning of any statement A^{abcd}, however high its degree, in which any of the exponents a, b, c, &c. may denote ϵ or η or θ, or any numerical fractions between 0 and 1.

Some symbol, say $\delta\,\dfrac{A}{B}$, may be conveniently employed as an abbreviation for $\dfrac{A}{B} - \dfrac{A}{\epsilon}$, to denote the *dependence of A upon B*; that is to say, for the increase or diminution in the chance of A being true, when the *hypothesis B* is added to our constant *data* ϵ. For example, speaking of a person taken at random out of a certain number of persons, each exactly fifty years of age, let A denote the statement "He will attain the age of sixty," and let B denote the statement "He is a baker." Let $\dfrac{A}{\epsilon} = a$, and let $\dfrac{B}{\epsilon} = b$. Suppose, from reference to statistics, we have the data $\left(\dfrac{A}{B} = k\,\dfrac{A}{B'}\right)$, and we are required to find $\delta\,\dfrac{A}{B}$, which would be a measure of the healthiness (or the reverse) of the occupation of baker. Putting a', b', c', &c. for $1-a$,

$1-b$, $1-c$, &c., it is easy to show that $\frac{A}{B'} = \frac{a}{b'} - \frac{b}{b'}\frac{A}{B}$, so that, from the data $\left(\frac{A}{B} = k\frac{A}{B'}\right)$, we get

$$\frac{A}{B} = \frac{ka}{1-b+kb};$$

$$\delta\frac{A}{B} = \frac{A}{B} - a = \frac{a(1-b)(k-1)}{1+b(k-1)}.$$

When $k=1$, then $\delta\frac{A}{B} = 0$, which would indicate that the occupation of baker neither increased nor diminished the chance of a man of fifty attaining the age of sixty. When $k>1$, then $\delta\frac{A}{B}$ is positive, and would measure the *increase* in the chance; whereas $k<1$ would make $\delta\frac{A}{B}$ negative, and we should have a measure of the *unhealthiness* of the occupation of baker for a man of the age in question.

Prof. Elliott has put at our disposal the following account of the Rev. Dr. Bartholomew Price:—

Bartholomew Price, D.D., F.R.S., F.R.A.S., Master of Pembroke College, Oxford, and Canon of Gloucester, who died December 29th, 1898, was son of the Rev. William Price, Rector of Farnborough, Wilts, and of Coln St. Denis, Gloucestershire, and was born at this latter village on May 14th, 1818. From the Grammar School of Northleach, near his home, he proceeded to Pembroke College as Scholar in 1837. He graduated with a First Class in Mathematics and a Third Class in Classics in 1840, was University Mathematical Scholar in 1842, and was elected Fellow of his college in 1844. This college he served at different times as Mathematical Lecturer, Tutor, Bursar, Vice-gerent (1864–92), and eventually as Master from 1892 till his decease. He was also Honorary Fellow of Queen's College from 1868 onwards. He was ordained deacon in 1841 and priest in 1843. For forty-five years (1853–98) he was Sedleian Professor of Natural Philosophy, resigning the chair, of which he was the twelfth occupant, in the last year of his life. His services to his University in other capacities were too great and varied for adequate description. He was many times Public Examiner and Moderator; was Proctor in 1858; Member of the Hebdomadal Council—the body which has *the initiative in* all academic legislation—continuously from 1856,

when it replaced the old Hebdomadal Board, till 1898; Curator of the Bodleian Library; Curator of the University Chest from the time when, very largely by himself, chaos was reduced to order in University finance till the end of his life; Secretary of the University Press from 1868 to 1884, and perpetual Delegate afterwards; Delegate of the Museum; Chairman of the Board of the Faculty of Natural Science from its foundation, in 1882, till he ceased to be Professor. Even this list of academical offices, in which his tact, knowledge of men, and extraordinary business gifts left their mark on the University, is not complete. To his organizing and administrative ability University and college finances, and the present prosperity of the Clarendon Press in particular, owe more than can be told. Nor can the quiet dignity and efficiency with which he presided as *doyen* of the Scientific Faculty be well over-rated.

He was one of the Visitors of the Royal Observatory, Greenwich, and one of the Governing Body of Winchester College. In 1872 he served on the Royal Commission, known as the Duke of Cleveland's Commission, for inquiring into the property and revenues of the Universities and Colleges of Oxford and Cambridge.

Price was a man of action and of affairs, a man of unsurpassed patience and coolness of judgment, and of the highest capacity to make the best use of men and opportunities, characterized by strenuous devotion to a cause in hand, quiet, methodical, and persistent, never carried by enthusiasm into rashness of action, always cautious in detail, but never swerving in principle or policy. Always busy to an extent which would produce the restlessness of over-work in ordinary men, he never showed haste. The trait was evident in small matters as in great. A man of much writing, his accustomed signature formed every letter of the lengthy Christian name Bartholomew. The speaker ever called upon when something important had to be enforced or something difficult to be made clear, he had to contend not only with the drawback of a low voice, but with an absolute difficulty of articulation, to which he never gave way. A pause would occur in his somewhat rapid utterance, but he would show no sign of flurry. The offending consonant would presently come out clear, and the smooth delivery of an effective sentence as begun would proceed.

He was elected a member of our Society on June 26th, 1866, but never contributed to our *Proceedings*. Indeed, though to the last year of his life he continued a Professor of Mathematics, and though he never gave up the study or the love of it or the service of its

interests, it may be said that the period when mathematical activity was to the forefront in his life had passed almost before the Society came into existence. But his services to mathematics were none the less real and lasting. Few teachers of mathematics have been so energetic or successful as he was in his young days, when well-nigh all the mathematical teaching of Oxford rested on him. As since, in developing the resources of great institutions which have been his care, so then, in finding out the strength of students and passing them on to success in life and thought, he was ever alive to facts, ever stimulating, and ever far-seeing. He found a lamentable want of worthy text-books in English, and set to work to supply the need by a treatise on a large scale. Mathematics has moved since then; and Price's four volumes on Infinitesimal Calculus and its Applications are probably little read now. But they did no small work in their time. They collected and arranged for the English mathematical student much of the best then recent work to be found in foreign books and memoirs, in a way which we are apt to associate with much later times, and the skill of the exposition shown in them was admirable. For lucidity of style they have not been surpassed, and but rarely equalled. Bohlmann (*Jahresbericht*, 1899) classes Price with Serret as a follower of Cauchy. The date of the first edition of the Differential Calculus was 1848. Expanded, it appeared as the first volume of his extended treatise in 1852, and was followed by the other volumes in 1854 (Integral Calculus), 1856 (Statics, Attractions, &c.), and 1862 (Dynamics of Material Systems).

He married in 1847 Amy daughter of W. C. Cole, Esq., of Exmouth, who survives him, as do almost all of a large family.

He was a man of simple tastes and of genial disposition, somewhat above middle height, of spare frame and homely features, with an expression watchful of all but unfriendly to none.

For the accompanying sketch of the late Prof. Sophus Lie's mathematical work we are indebted to Prof. W. Burnside.

By the death on February 18th, 1899, of Sophus Lie, the Society has lost one of its most distinguished foreign members.

Sophus Lie was born in December, 1842, at a small village near Florö in Norway, where his father was the minister. His mathematical genius does not seem to have shown itself in his early years, as, even when about to leave the University of Christiania in 1865, he was still doubtful as to his vocation. Within two or three years *of that time,* however, his mind must have been made up; for in

1869 his earliest memoir "Repräsentation der Imaginären in der Plan-geometrie," was published in the *Verhandlungen der Gesellschaft der Wissenschaft zu Christiania.* Soon after this Lie and his friend Klein were fellow-students at Paris, studying the theory of groups of substitutions, on which subject M. Jordan was then lecturing.

In 1872 Lie published his first famous memoir, "Ueber Complexe, insbesondere Linien- und Kugel-complexe, mit Anwendungen auf die Theorie partiellen Differentialgleichungen" in the *Math. Annalen.* This memoir contains some of the author's earliest ideas on contact-transformations; and establishes a very remarkable correspondence between systems of straight lines and systems of spheres. In 1873 appeared Lie's first detailed account of contact-transformations (*Christiania Verhandlungen*), and in 1874 he gave to the world his earliest sketch of the theory of continuous groups (*Christiania Verhandlungen* and *Göttingen Nachrichten*), with which, in all probability, his name will in the future be most closely connected. For ten or twelve years onwards from this date Lie published memoirs in considerable number and of great importance, further developing the theories of point- and of contact-transformations and that of continuous groups, with many applications both geometrical and analytical; and others dealing with several of the most interesting problems of geometry, especially those concerned with minimum surfaces and with surfaces of constant curvature.

In the earlier seventies Lie was appointed to an Extraordinary Professorship of Mathematics at Christiania, and this he held till 1886, when he accepted an invitation to become Professor at Leipzig, Klein having just been called to Göttingen. It was at Leipzig that the memoirs on non-Euclidean geometry and on the axioms that lie at the base of all geometry were written. Here, too, he undertook the work of throwing the results of his earlier researches into a more directly didactic form. The three volumes of the *Theorie der Transformationsgruppen*, in the preparation of which he was assisted by Dr. F. Engel, appeared in 1888, 1890, and 1893. In 1893 appeared also the less elaborate *Vorlesungen über continuierliche Gruppen mit geometrischen und anderen Anwendungen*, which had been preceded in 1891 by the *Vorlesungen über gewöhnliche Differentialgleichungen mit bekannten infinitesimalen Transformationen.* Both of the latter works were edited by Dr. G. Scheffers. In 1896 was published the first volume of the *Geometrie der Berührungstransformationen*, in the production of which Dr. Scheffers was again associated with Lie. The complete work was to have consisted of two volumes. [illegible] on

the theory of infinite continuous groups, written in collaboration with Dr. Engel, was also promised, but it unfortunately has not appeared.

Last summer Lie returned to Christiania to occupy a chair which had been specially created for him by the Norwegian Parliament. He hardly lived, however, to take effective possession of it, for excessive work had undermined his health and strength. He died, as already stated, on February 18th of the present year, from the effects of cerebral anæmia, at the age of fifty-six.

Lie is probably best known in this country by his three volumes, written in conjunction with Dr. Engel, on the theory of transformation groups. In this work an entirely new theory, which has already had an extraordinary influence on the advance of mathematical science, is developed with wonderful completeness, while many of its applications are investigated in great detail. Yet it is nearly certain that most English readers must find the book wearisome. Lie, in fact, is not seen in it in his true light. Klein's advice, in the first of his lectures on Lie in the Evanston Colloquium, may be quoted as a finger-guide to those who would appreciate Lie correctly:—

"To fully understand the mathematical genius of Sophus Lie, one must not turn to the books recently published by him in collaboration with Dr. Engel, but to his earlier memoirs written during the first years of his scientific career. There Lie shows himself the true geometer he is, while in his later publications, finding that he was but imperfectly understood by the mathematicians accustomed to the analytical point of view, he adopted a very general analytical form of treatment that is not always easy to follow."

Sophus Lie was, in fact, above all things, a great geometer; and the present notice may fitly close with a quotation from his own writings, indicating how constantly he kept the geometrical side of his investigations before his mind:—

"In meinen wissenschaftlichen Bestrebungen bin ich immer von der Auffassung ausgegangen, dass es . . . wünschenswert ist, dass sich Analysis und Geometrie ebenso wie früher auch in unserer Zeit gegenseitig stützen und mit neuen Ideen bereichern."

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